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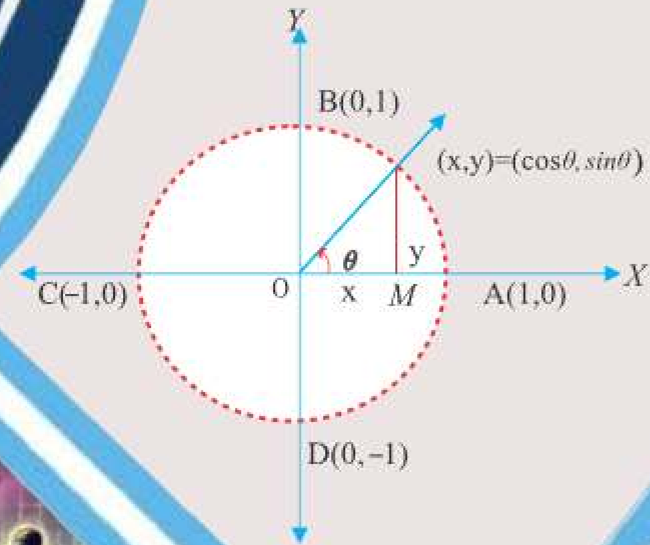

MATHEMATICS

PUNJAB EDUCATION, CURRICULUM, TRAINING
AND ASSESSMENT AUTHORITY

بِسۡمِ اللّٰهِ الرَّحۡمٰنِ الرَّحِيۡمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

MATHEMATICS

11

PUNJAB EDUCATION, CURRICULUM, TRAINING
AND ASSESSMENT AUTHORITY

Compilers:

- Prof. Shamhad Muhammad Lodhi (late)
- Prof. Muhammad Sharif Ghori (late)
- Prof. Sanaullah Bhatti (late)
- Prof. Khalid Saleem
- Prof. Muhammad Amin Chaudhary

Editor:

- Majid Hameed, Master Trainer PEF, Lahore

(Authors) Aligned with Revised/ Updated NCP-2023 by:

- Dr. Malik Anjum Javeed, HED
- Dr. Muhammad Idrees, HED
- Muhammad Akhtar Shirani, SSS PECTTA
- Ghulam Murtaza, SSS PECTTA
- Madiha Mehmood, SS PECTTA

ERC Members:

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- Dr. Irfan Ahmad Aslam, PEF, Lahore
- Talmeez-ur-Rehman (SS), SED, Lahore

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Supervised by:

- Muhammad Akhtar Shirani
- Ghulam Murtaza
- Madiha Mehmood

Deputy Director (Graphics):

- Aisha Sadiq

Designer:

- Atif Majeed
- Kamran Afzal Butt

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Unit 1 Complex Numbers

INTRODUCTION

Complex numbers are an extension of the real numbers designed to solve equations that have no solutions within the realm of real numbers. The history of mathematics shows that man has been developing and enlarging his concept of **number** according to the saying that "Necessity is the mother of invention". In the remote past they started with the set of counting numbers and invented, by stages, the negative numbers, rational numbers, irrational numbers etc. Since square of a positive as well as negative number is a positive number, the square root of a negative number does not exist in the realm of real numbers. Therefore, square roots of negative numbers were given no attention for centuries together. However, recently, properties of numbers involving square roots of negative numbers have also been discussed in detail and such numbers have been found useful and have been applied in many branches of pure, applied, financial and computational mathematics.

1.1 Complex Numbers

The numbers of the form $z = a + ib$ where $a, b \in \mathcal{R}$ and $i = \sqrt{-1}$, are called **complex numbers.** For example, $3 + 4i$, $2 - \frac{5}{7}i$, $-7-2i$ etc. are complex numbers and the set of all complex numbers is denoted by C

1.1.1 Recognition of Real and Imaginary Parts

Let us start with considering the following equation:

$$x^2 + 1 = 0 \quad \Rightarrow \quad x^2 = -1 \quad \Rightarrow \quad x = \pm\sqrt{-1}$$

$\sqrt{-1}$ does not belong to the set of real numbers. We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota).

Note:
Every real number is a complex number with 0 as its imaginary part.

In the complex number $z = a + ib$, a is called **real part** and b is called **imaginary part** of the complex number. For convenient, real part is denoted by Re z and imaginary part by Im z of a complex number z. For example, if $z = 3 + 4i$, then

$$\text{Re } z = 3 \text{ and Im } z = 4.$$

The product of a non-zero real number and i is also an **imaginary number** and is written as 1. Thus $2i, -3i, \sqrt{5}i, -\frac{11}{2}i$ are all imaginary numbers.

Conjugate Complex Numbers: Let $z = a + ib$ be a *complex number,* then $a - ib$ is called the complex conjugate of $a + ib$. It is denoted by $\overline{z}$. Thus $5 - 4i$ is complex conjugate of $5 + 4i$ and $-2 - 3i$ is complex conjugate of $-2 + 3i$.

Note: A real number is self-conjugate.

1.1.2 Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions.

The symbols *a, b, c, d, k,* where used, represent real numbers.

(i) Addition: $(a+ib)+(c+id)=(a+c)+i(b+d)$

(ii) $k(a+ib)=ka+ikb$

(iii) Subtraction: $(a+ib)-(c+id) = (a+ib)+[-(c+id)]$

$$= a+ib+(-c-id)=(a-c)+i(b-d)$$

(iv) Multiplication: $(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$

1.1.3 Complex Numbers as Ordered Pairs of Real Numbers

We can define complex numbers also by using ordered pairs.

Let C be the set of ordered pairs belonging to $\mathbb{R} \times \mathbb{R}$ which are subject to the following properties:

(i) $(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d$

(ii) $(a, b) + (c, d) = (a + c, b + d)$

(iii) $(a, b)(c, d) = (ac - bd, ad + bc)$

Then C is called the set of **complex numbers**. It is easy to see that

$$(a, b) - (c, d) = (a - c, b - d)$$

(iv) If k is any real number, then $k(a,b) = (ka, kb)$

Properties (i), (ii) and (iii) respectively define equality, sum and difference of two complex numbers. Property (iv) defines the product of a real number and a complex number.

Example 1: Find the sum, difference and product of the complex numbers (8, 9) and (5, –6)

Solution: Sum = (8 + 5, 9 – 6) = (13, 3)

Difference = (8 – 5, 9 – (– 6)) = (5, 15)

Product = $(8\cdot 5-(9)(-6),\ 9\cdot 5+(-6)(8))$

= (40 + 54, 45 – 48) = (94, –3)

1.1.4 Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties of real numbers.

By way of explanation of some points we observe as follows:

(i) The additive identity in C is (0, 0).

(ii) Every complex number (a, b) has the additive inverse $(-a, -b)$ i.e.,
$(a, b) + (-a, -b) = (0, 0)$

(iii) The multiplicative identity is (1, 0) i.e.,

$$(a, b)\cdot(1, 0) = (a\cdot 1 - b\cdot 0,\ b\cdot 1 + a\cdot 0) = (a, b).$$
$$= (1, 0)\cdot(a, b)$$

(iv) Every non-zero complex number {i.e., number not equal to (0,0)} has a multiplicative inverse. The multiplicative inverse of (a, b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$$\because \quad (a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0), \text{ the identity element}$$
$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a, b)$$

(v) $(a, b)\,[(c, d) \pm (e, f)] = (a, b)(c, d) \pm (a, b)(e, f)$

Note:
The set C of complex numbers does not satisfy the order axioms. In fact, there is no sense in saying that one complex number is greater or less than the other.

Example 2: If $z_1 = (4, 2)$ and $z_2 = (3, -1)$, then find $\frac{z_1}{z_2}$.

Solution: Given $z_1 = (4, 2)$, $z_2 = (3, -1)$

$$\text{Now, } \frac{z_1}{z_2} = \frac{(4, 2)}{(3, -1)} = \frac{4+2i}{3-i}$$

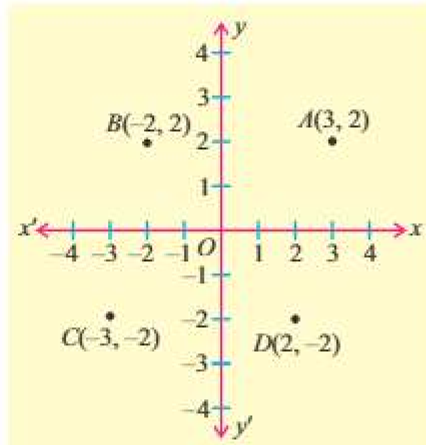
Multiply the numerator and denominator by the complex conjugate of $z_2 = 3 - i$.

$$\frac{z_1}{z_2} = \frac{4+2i}{3-i} = \frac{4+2i}{3-i} \times \frac{3+i}{3+i}$$
$$= \frac{(4)(3)+(4)(i)+(2i)(3)+(2i)(i)}{(3)^2-(i)^2} = \frac{12+4i+6i+2i^2}{9-i^2}$$
$$= \frac{12+10i-2}{9-(-1)} = \frac{10+10i}{10} = 1+i \qquad \because \quad i^2 = -1$$

$$\text{Thus, } \frac{z_1}{z_2} = 1+i$$

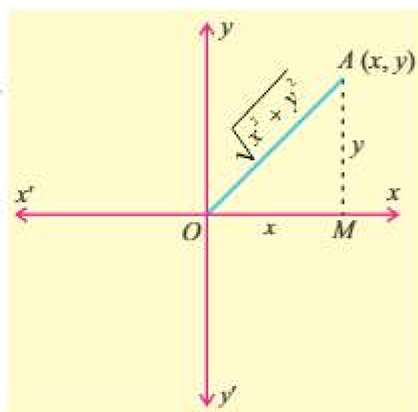
1.1.5 Argand Diagram

Every complex number will be represented by one and only one point of the coordinate plane and every point of the plane will represent one and only one complex number. The components of the complex number will be the coordinates of the point representing it. In this representation the x-axis is called the real axis and the y-axis is called the imaginary axis. The coordinate plane itself is called the complex plane or z – plane. The figure representing one or more complex numbers on the complex plane is called an Argand diagram. The Argand diagram is a way of representing one or more complex numbers on the complex plane. Points on the x-axis represent real numbers whereas the points on the y-axis represent imaginary numbers.



In an Argand diagram, the complex number $x+iy$ is uniquely represented by the order pair (x, y). In Figure (i), the complex numbers $3 + 2i$, $-2 + 2i$, $-3 - 2i$ and $2 - 2i$ correspond to the order pairs (3, 2), (−2, 2), (−3, −2) and (2, −2) respectively have been represented geometrically by the point A, B, C and D.

Modulus of Complex Number: The real number $\sqrt{x^2+y^2}$ is called the modulus of the complex number $x+iy$ and it is denoted by $|x+iy|$.In Figure (ii), $|\overline{OA}|$ represent the modulus of $x+iy$. In other words, the modulus of a complex number is the distance from the origin to the point representing the number.



Example 3: If $z = \dfrac{(1+2i)^2}{2-i}$ then evaluate $|\bar{z}|$

Solution:

$$z = \frac{(1+2i)^2}{2-i} = \frac{1+4i+4i^2}{2-i} = \frac{-3+4i}{2-i} \times \frac{2+i}{2+i} = \frac{-6-3i+8i+4i^2}{2^2-i^2}$$

$$= \frac{-6+5i-4}{4-(-1)} = \frac{-10+5i}{5}$$

$$\Rightarrow \quad z = -2+i$$

Taking conjugate

$$\bar{z} = \overline{-2-i} = -2+i$$

and $|\bar{z}| = |-2-i| = \sqrt{(-2)^2+(-1)^2} = \sqrt{4+1}$

$\Rightarrow \quad |\bar{z}| = \sqrt{5}$

EXERCISE 1.1

1. Simplify the following:

(i) i^9 (ii) i^{17} (iii) $(-i)^{19}$ (iv) $(-1)^{\frac{-21}{2}}$

2. Prove that $\bar{z} = z$ iff z is real.
3. For $z \in C$, show that:

(i) $\dfrac{z+\bar{z}}{2} = \text{Re}(z)$ (ii) $\dfrac{z-\bar{z}}{2i} = \text{Im}(z)$

(iii) $|z|^2 = z \cdot \bar{z}$ (iv) $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

4. Find the multiplicative inverse of each of the following numbers:

(i) $(-4, 7)$ (ii) $(\sqrt{2}, -\sqrt{5})$ (iii) $(1,0)$

5. Separate into real and imaginary parts (write as a simple complex number):

(i) $\dfrac{2-7i}{4+5i}$ (ii) $\dfrac{(-2+3i)^2}{1+i}$ (iii) $\dfrac{i}{1+i}$ (iv) $\dfrac{(4+3i)^2}{4-3i}$

6. If $z_1 = 2+i, z_2 = 3-2i, z_3 = 1+3i$ then express $\dfrac{\overline{z_1}\,\overline{z_3}}{z_2}$ in the form of $a+ib$.

7. If $z_1 = 2+7i$ and $z_2 = -5+3i$, the evaluate the following:

(i) $|2z_1 - 4z_2|$ (ii) $|3z_1 + 2\overline{z_1}|$ (iii) $|-7z_2 + 2\overline{z_2}|$ (iv) $|(z_1+z_2)^3|$

1.2 Equality of Two Complex Numbers

The two complex numbers $z_1 = a+bi$ and $z_2 = c+di$ are said to be equal iff their real and imaginary parts are equal i.e., $a+bi = c+di \Leftrightarrow a = c$ and $b = d$.

Example 4: If $(3+2i)(x+iy) = 5+12i$, where $x, y \in R$, then find the values of x and y.

Solution: Given that $(3+2i)(x+iy) = 5+12i$

$\Rightarrow \quad 3x + 3iy + 2ix + 2yi^2 = 5 + 12i$

$\Rightarrow \quad (3x-2y) + (2x+3y)i = 5 + 12i$

Comparing real and imaginary part, we have

$$3x - 2y = 5 \qquad \ldots(i)$$
$$2x + 3y = 12 \qquad \ldots(ii)$$

Multiplying equation (i) by 3 and equation (ii) by 2, we have

$$9x - 6y = 15$$
$$4x + 6y = 24$$

Add the equations

$$9x - 6y + 4x + 6y = 15 + 24$$
$$13x = 39$$
$$x = 3$$

Substitute $x = 3$ in equation (i), we have

$$9(3) - 6y = 15$$
$$-6y = -12$$
$$y = 2$$

Thus, $x = 3, y = 2$

1.2.1 Square Root of a Complex Number

The square root of a complex number is another complex number that, when squared, give the original complex number.

Let $w = p + qi$ is a square root of a complex number $z = x + iy$, where $p, q, x, y \in R$, then $w = \sqrt{z}$ …(i), taking square on both sides, we get

$$w^2 = z$$
$$(p + iq)^2 = x + iy$$
$$p^2 + 2pqi - q^2 = x + iy$$

Equating real and imaginary part, we have

$$x = p^2 - q^2 \qquad \ldots(ii)$$
$$y = 2pq \qquad \ldots(iii)$$

We know that $(p^2 + q^2)^2 = (p^2 - q^2)^2 + 4p^2q^2$

Substitute $x = p^2 - q^2$, $y = 2pq$ in the above equation, we get

$$(p^2 + q^2)^2 = x^2 + y^2$$
$$\Rightarrow \quad p^2 + q^2 = \sqrt{x^2 + y^2} \qquad \ldots(iv)$$

From equation (ii) and (iv), we have $x = p^2 - q^2$ and $p^2 + q^2 = \sqrt{x^2 + y^2}$. Solving for the values p and q, we have

$$p = \pm\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} \quad \text{and} \quad q = \pm\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}$$

From equation (iii): $y = 2pq$, thus we have

- $y > 0$, if p and q have the same sign
- $y < 0$, if p and q have opposite sign
- $y = 0$, if $p = 0$ or $q = 0$

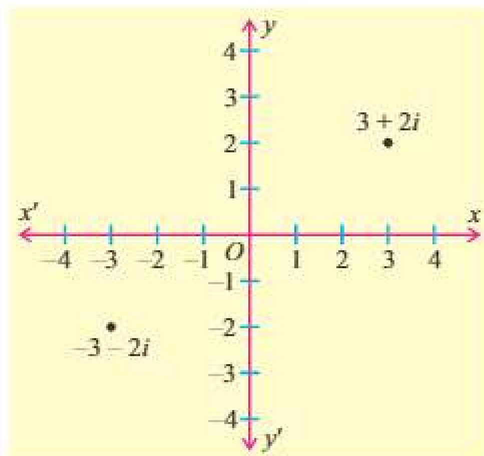
Therefore, the square root of the complex number $z = x + iy$ is given by

$$\sqrt{z} = \sqrt{x+iy} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$$

or $\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$ ...(v), where $|z| = \sqrt{x^2+y^2} \geq 0$ is modulus of z.

equation (v) is the required formula for square root of complex numbers.

Example 5: Find the square root of complex number 5 + 12i and also represent the square roots on an Argand diagram.



Solution: Let $x + yi = 5 + 12i$

$$\Rightarrow x = 5 \text{ and } y = 12 > 0$$

$$|z| = |5+12i| = \sqrt{5^2+12^2} = 13,$$

Applying the square root formula for complex numbers, we get

$$\sqrt{5+12i} = \pm\left(\sqrt{\frac{13+5}{2}} + \frac{i12}{|12|}\sqrt{\frac{13-5}{2}}\right)$$

$$= \pm\left(\sqrt{9} + i\sqrt{4}\right) = \pm(3+2i)$$

Thus, the square root of the complex number 5 + 12i are 3 + 2i and −3 − 2i are shown in adjacent figure.

EXERCISE 1.2

1. Find the values of x and y in each of the following:

(i) $x + iy + 2 - 3i = i(5-i)(3+4i)$

(ii) $(x + iy)(1 - i) = (2 - 3i)(-5 + 5i)(-i3/5)$

(iii) $\frac{x}{2+i} + \frac{y}{3-i} = 4 + 5i$

2. If $z_1 = -13 + 24i$ and $z_2 = x + yi$, find the values of x and y such that $z_1 - z_2 = -27 + 15i$
3. Find the value of x and y if:

   (i) $(x+iy)^2 = 25 + 60i$ (ii) $(x+iy)^2 = 64 + 48i$ (iii) $x + iy = \dfrac{-2-5i}{(1+3i)^3}$
4. If $z_1 = 2 + 3i$ and $z_2 = 1 - \alpha$, find the value of α such that $\text{Im}(z_1 z_2) = 7$.
5. If $z_1 = x + yi$ and $z_2 = a + bi$, find x, y, a and b such that $z_1 + z_2 = 10 + 4i$ and $z_1 - z_2 = 6 + 2i$.
6. Show that $\forall\, z_1, z_2 \in C$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
7. Find the square root of the following complex numbers:

   (i) $-7 - 24i$ (ii) $8 - 6i$ (iii) $-15 - 36i$ (iv) $119 + 120i$
8. Find the square root of $13 - 20\sqrt{3}i$ and represent them on an Argand diagram.
9. Find the value of x and y if $(-7+i)(x+iy) + (-1-5i) = i(11-i)$
10. Find the value of x and y if $(5-2i)(x+iy) + 3 = i(11-i) - 4i$
11. Find the values of u and v if:

    (i) $(u+iv)^2 = 20 + 21i$ (ii) $(u+iv)^2 = 48 - 10i$
12. If $z_1 = 4 + 5i$ and $z_2 = \alpha - 2i$, find the value of α such that $\text{Re}(z_1 z_2) = 20$.

1.2 Complex Polynomials as a Product of Linear Factors

A **complex polynomial** $P(z)$ is a polynomial function of the complex variable z with complex coefficients. It is expressed in the general form as

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

Where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers $(a_n \neq 0)$, and $n \geq 0$ is an integer representing the degree of the polynomial.

For examples $P_1(z) = (1-i)z + 3i$, $P_2(z) = (5-4i)z^2 + (2+i)z + (3-4i)$ and $P_3(z) = (2-i)z^3 + 2z^2 i + (5+3i)$ are the examples of linear, quadratic and cubic complex polynomials respectively. If $n = 0$, then $P(z)$ becomes a constant polynomial. A fundamental property of complex polynomials is that they can always be factored into a product of linear factors.

According to the **Fundamental theorem of algebra**, a polynomial of degree $n \geq 1$ has exactly n roots in complex numbers system C.

A **corollary** to this theorem states that any polynomial $P(z)$ of degree n can be factored completely into a constant a and n linear factor over C in the form

$$P(z) = a(z - z_1)(z - z_2)...(z - z_n) \qquad (1)$$

where $z_1, z_2, ..., z_n$ are complex roots of the polynomial. Once we know the roots of a polynomial equation, we can apply equation (1) to factored the polynomial $P(z)$ into n linear factors. Specifically, if z_1 and z_2 are roots of the polynomial equation $P(z)$, then the equation must be $P(z) = (z - z_1)(z - z_2)$. For examples, the polynomial $P(x) = x^2 + 4$ consists of real coefficient has no real roots, so it cannot be factored into linear polynomials with real coefficients. However, if we considered as a complex polynomial $P(z) = z^2 + 4$, we can easily be factored into two linear factors as

$$z^2 + 4 = (z + 2i)(z - 2i)$$

where $2i$ and $-2i$ are the complex roots of $z^2 + 4 = 0$

Note: If $P(z)$ is a polynomial function, the values of z that satisfy $P(z) = 0$ are called the zeros of the function $P(z)$ and roots of the polynomial equation $P(z) = 0$.

Example 6: Factorize the polynomial $P(z) = z^2 + (1 - i)z - i$.

Solution:

$$\begin{aligned} P(z) &= z^2 + (1 - i)z - i \\ &= z^2 + z - iz - i \\ &= z(z + 1) - i(z + 1) \\ &= (z + 1)(z - i) \end{aligned}$$

Example 7: Factorize the polynomial $P(z) = z^2 - 4iz + 12$

Solution:

$$\begin{aligned} P(z) &= z^2 - 4iz + 12 \\ &= z^2 - 4iz - (-12) \\ &= z^2 - 4iz - i^2 12 \qquad \because \quad i^2 = -1 \\ &= z^2 - i6z + i2z - i^2 12 \\ &= z(z - 6i) + 2i(z - 6i) \\ &= (z - 6i)(z + 2i) \end{aligned}$$

Example 8: Factorize the polynomial $P(z) = z^3 + (1 + i)z^2 + iz$.

Solution:

$$\begin{aligned} P(z) &= z^3 + (1 + i)z^2 + iz \\ &= z[z^2 + (1 + i)z + i] \\ &= z[z^2 + z + iz + i] \\ &= z[z(z + 1) + i(z + 1)] \\ &= z[(z + 1)(z + i)] \\ &= z(z + 1)(z + i) \text{ are linear factors.} \end{aligned}$$

Key Concept

The Rational Root Theorem is a mathematical tool used to find all possible rational roots of a polynomial equation with integer coefficients. According to rational root theorem:

If a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ has integer coefficients, then every rational root $\frac{p}{q}$ (in simplest terms) satisfies:

(i) p is a factor of the constant term a_0. (ii) q is a factor of the leading coefficient a_n.

Example 9: Factorize the + polynomial $P(z) = z^3 - 3z^2 + z + 5$.

Solution: According to rational root theorem the possible root of the equation are ± 1 and ± 5. On checking, we see that $z = -1$ is the root of the polynomial $P(z)$ because

$$P(-1) = (-1)^3 - 3(-1)^2 + (-1) + 5 = 0.$$

So $z + 1$ is a factor of the $P(z)$. Using synthetic division

−1	1	−3	1	5
		−1	4	−5
	1	−4	5	0

Therefore, $z^3 - 3z^2 + z + 5 = (z+1)(z^2 - 4z + 5)$ ...(i)

Next find the factors of $z^2 - 4z + 5$ using quadratic formula

$z^2 - 4z + 5 = 0$, here $a = 1, b = -4, c = 5$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$

$$\Rightarrow \quad z = 2 \pm 2i$$

The quadratic factors are $z^2 - 4z + 5 = (z - (2+i))(z - (2-i)) = (z - 2 - i)(z - 2 + i)$

Substitutes in equation (i), we have the

$$z^3 - 3z^2 + z + 5 = (z+1)(z-2-i)(z-2+i)$$

1.3.1 Solution of Quadratic Equations by Completing the Square

As we learned in previous classes, **completing the square** is a powerful and systematic method for solving quadratic equations. This technique involves rewriting a quadratic equation in the form $ax^2 + bx + c = 0$ into a perfect square trinomial, which can then be solved by taking the square root of both sides. This method is especially valuable when the quadratic equation does not factor easily. By completing the square, we can solve any quadratic equation, even those with irrational or complex roots, making it a more effective technique in algebra.

Example 10: Solve the equation $2z^2 - 12z + 50 = 0$ by completing square method and hence express it as a product of its linear factors.

Solution: $2z^2 - 12z + 50 = 0$

Dividing both sides by 2

$$z^2 - 6z + 25 = 0$$

$$\Rightarrow \quad z^2 - 2(3)z = -25$$

Add 3^2 on both sides

$$z^2 - 2(3)z + 3^2 = -25 + 3^2$$

$$(z-3)^2 = -16$$

$$\Rightarrow \quad z - 3 = \pm\sqrt{-16}$$

$$\Rightarrow \quad z = 3 \pm 4i$$

Therefore, $z = 3 + 4i$ or $z = 3 - 4i$ are the required complex roots.

Using the corollary of Fundamental theorem of Algebra the equation can be factorized using the roots $3 + 4i$ and $3 - 4i$ as:

$$2z^2 - 12z + 50 = 2(z^2 - 6z + 25) = 2\left(z-(3+4i)\right)\left(z-(3-4i)\right) = 2\left(z-3-4i\right)\left(z-3+4i\right)$$

Hence, $2z^2 - 12z + 50 = 2\left(z-3-4i\right)\left(z-3+4i\right)$

EXERCISE 1.3

1. Factorize the following:

   (i) $a^2 + 4b^2$ (ii) $9a^2 + 16b^2$ (iii) $3x^2 + 3y^2$ (iv) $144x^2 + 225y^2$

   (v) $z^2 - 2iz - 1$ (vi) $z^2 + 6z + 13$ (vii) $z^2 + 4z + 5$ (vii) $2z^2 - 22z + 65$

2. Factorize the following polynomial into its linear factors:

   (i) $z^3 + 8$ (ii) $z^3 + 27$ (ii) $z^3 - 2z^2 + 16z - 32$ (iii) $z^4 + 21z^2 - 100$

   (iv) $z^4 - 16$ (iii) $z^4 + 3z^2 - 4$ (iv) $z^4 + 5z^2 + 6$ (v) $z^4 + 7z^2 - 144$

3. Find the roots of $z^4 + 7z^2 - 144 = 0$ and hence express it as a product of linear factors.

4. Solve the following complex quadratic equation by completing square method:

   (i) $2z^2 - 3z + 4 = 0$ (ii) $z^2 - 6z + 30 = 0$ (iii) $3z^2 - 18z + 50 = 0$

   (i) $z^2 + 4z + 13 = 0$ (ii) $2z^2 + 6z + 9 = 0$ (iii) $3z^2 - 5z + 7 = 0$

5. Solve the following equations:

   (i) $2z^4 - 32 = 0$ (ii) $3z^5 - 243z = 0$ (iii) $5z^5 - 5z = 0$

   (iv) $z^3 - 5z^2 + z - 5 = 0$ (v) $4z^4 - 25z^2 + 21 = 0$ (vi) $z^3 + z^2 + z + 1 = 0$

6. Find a polynomial of degree 3 with zeros 3, $-2i$, $2i$ and satisfying $P(1) = 20$.

7. Find a polynomial of degree 4 with zeros $2i$, $-2i$, 1, -1, and satisfying $P(2) = 240$.

8. Find a polynomial of degree 4 with zeros 4, -4, $1 + i$, $1 - i$ and satisfying $P(2) = 72$.

1.4 Three Cube Roots of Unity

Let x be a cube root of unity

$$\therefore \quad x = (1)^{\frac{1}{3}}$$

$$\Rightarrow \quad x^3 = 1$$

$$\Rightarrow \quad x^3 - 1 = 0$$

$$\Rightarrow \quad (x-1)(x^2+x+1) = 0$$

Either $\quad x - 1 = 0 \Rightarrow \quad x = 1$

or $\quad x^2 + x + 1 = 0$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{3}i}{2} \qquad (\because \sqrt{-1} = i)$$

Note:

We know that the numbers containing i are called **Complex numbers**. So $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

Thus, the three cube roots of unity are:

$$1, \quad \frac{-1+\sqrt{3}i}{2} \text{ and } \frac{-1-\sqrt{3}i}{2}$$

1.4.1 Properties of Cube Roots of Unity

(i) Each complex cube root of unity is square of the other

If $\frac{-1+\sqrt{3}i}{2} = \omega$, then $\frac{-1-\sqrt{3}i}{2} = \omega^2$,

and if $\frac{-1+\sqrt{3}i}{2} = \omega$, then $\frac{-1+\sqrt{3}i}{2} = \omega^2$ [ω is read as omega]

(ii) The sum of all the three cube roots of unity is zero i.e., $1 + \omega + \omega^2 = 0$

(iii) The product of all the three cube roots of unity is unity i.e., $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$

1.5 Four Fourth Roots of Unity

Let x be the fourth root of unity

$$\therefore \quad x = (1)^{\frac{1}{4}}$$

$$\Rightarrow \quad x^4 = 1$$

$$\Rightarrow \quad x^4 - 1 = 0$$

$$\Rightarrow \quad (x^2-1)(x^2+1) = 0$$

$$\Rightarrow \quad x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

and $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i$.

Hence four fourth roots of unity are: $1, -1, i, -i$.

1.5.1 Properties of four Fourth Roots of Unity

We have found that the four fourth roots of unity are: $1, -1, +i, -i$

(i) Sum of all the four fourth roots of unity is zero

$\because \quad 1 + (-1) + i + (-i) = 0$

(ii) The real fourth roots of unity are additive inverses of each other
1 and -1 are the real fourth roots of unity and $1 + (-1) = 0 = (-1) + 1$

(iii) Both the imaginary fourth roots of unity are conjugate of each other
i and $-i$ are imaginary fourth roots of unity, which are obviously conjugates of each other.

(iv) Product of all the fourth roots of unity is -1 i.e., $1 \times (-1) \times i \times (-i) = -1$

Example 11: Prove that: $(x^3 + y^3) = (x + y)(x + \omega y)(x + \omega^2 y)$

Solution: R.H.S $= (x + y)(x + \omega y)(x + \omega^2 y)$

$= (x + y)[x + (\omega + \omega^2)yx + \omega^3 y^2]$

$= (x + y)(x^2 - xy + y^2) = x^3 + y^3 \quad \{\because \omega^3 = 1, \omega + \omega^2 = -1\}$ = L.H.S.

Hence proved.

EXERCISE 1.4

1. Find the three cube roots of:

(i) 8 (ii) -8 (iii) -27 (iv) 64 (v) -625

2. Evaluate:

(i) $(1 + \omega - \omega)^8$ (ii) $\omega^{28} + \omega^{29} + 1$ (iii) $(1 + \omega - \omega^2)(1 - \omega + \omega^2)$

(iv) $\left(\frac{1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$ (v) $(1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5$

3. Show that: (i) $x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y)$

(ii) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

(iii) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ $2n$ factors $= 1$

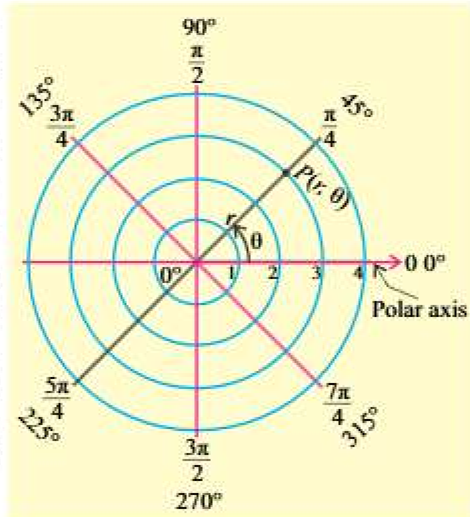
4. If ω is a root of $x^2 + x + 1 = 0$, show that its other root is ω^2 and hence prove that $\omega^3 = 1$.

5. Prove that complex cube roots of -1 are $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$; and hence prove that $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9 = -2$. Prove that $(-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 = 16$

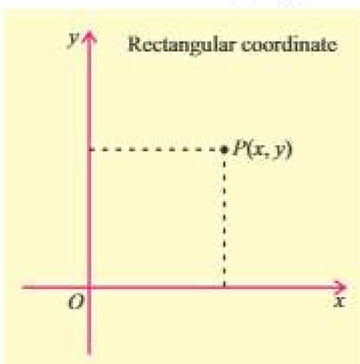
6. If ω is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$.

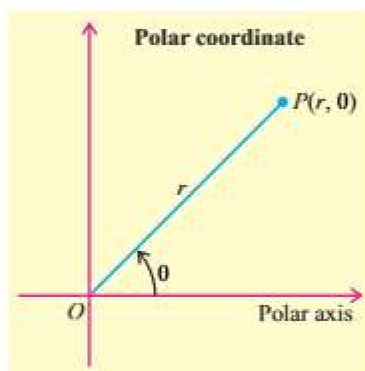
1.4 Polar Coordinates System

Polar coordinates are often more convenient than Cartesian coordinates in situations involving circular or rotational symmetry, or when a problem depends on distance from a fixed point and angle relative to a reference direction. Just as the Cartesian coordinate system uses an ordered pair (x, y) to describe the position of a point, the polar coordinate system determines the position of a point using a directed distance r from a fixed origin O (called the pole) and an angle θ that the line connecting the origin to the point makes with the polar axis (typically aligned with the positive x-axis).



In polar coordinate system the location of a point P can be described by polar coordinates in the form (r, θ), where r and θ are real numbers.

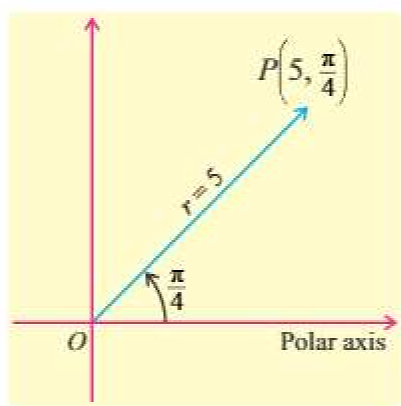




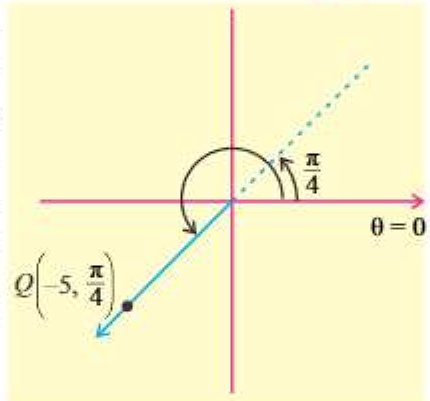
While r is typically considered non-negative $(r \geq 0)$, it is also possible for r to be negative $(r < 0)$. The value of r changes depending on its sign, and this affects the position of the point in the plane.

When $r > 0$, the angle θ is the measure of any angle in standard position whose terminal side lies along the line connecting the origin to the point P, measured counter clockwise from the polar axis (positive x-axis).

For example, the polar coordinates $\left(5, \frac{\pi}{4}\right)$ represent a point 5 units away from pole at an angle of $\frac{\pi}{4}$ radians.



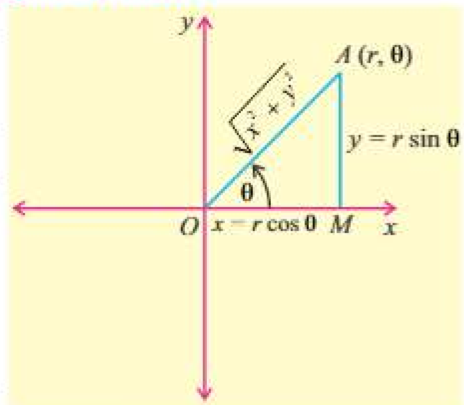
When $r < 0$, the angle θ is the measure of any angle in standard position whose terminal side lies along the line connecting the origin to the point Q, but the point Q is located $|r|$ units in the opposite direction (i.e., $\theta + \pi$) from the polar axis (positive x-axis). For example, the polar coordinates $\left(-5, \frac{\pi}{4}\right)$ represent a point 5 units away from the pole, but in the direction of $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$ radians.



Note: $(-5, \pi/4)$ and $(5, 5\pi/4)$ represent the same point in the plane

1.4.1 Polar Coordinate System of a Complex Number

Consider the adjoining diagram representing the complex number $z = x + iy$. From the diagram, we see that $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ is modulus and θ is called an argument of z.



Hence $\quad x + iy = r\cos\theta + i\,r\sin\theta \qquad$ (i)

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$ $(x \neq 0)$

Equation (i) is called the polar form of the complex number z.

Note: We can write $\cos\theta + i\sin\theta = \text{cis}\,\theta$

Example 12: Express the complex number $1 + i\sqrt{3}$ in polar form.

Solution: **Step – I** : Put $r\cos\theta = 1$ and $r\cos\theta = \sqrt{3}$

Step – II : $r^2 = (1)^2 + \left(\sqrt{3}\right)^2$

$\Rightarrow \quad r^2 = 1 + 3 = 4$

$\Rightarrow \quad r = 2$

Step – III : $\quad \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = 60^\circ$

Thus $\quad 1 + i\sqrt{3} = 2\cos 60^\circ + i\,2\sin 60^\circ$

Note:
- If $x = 0$, $y > 0$ then $\theta = \frac{\pi}{2}$
- If $x = 0$, $y < 0$ then $\theta = -\frac{\pi}{2}$
- If $x = 0$, $y = 0$ then θ is undefined.

Principal Argument: The principal argument θ of a complex number $z = a + bi$ is the angle between the positive real axis and the line joining (a, b) to the origin in the Argand plane.

$$arg\, z = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

It is denoted by *arg*. It is a single, specific value of the argument, typically chosen within a standard range: $arg\, z \in (-\pi, \pi]$.

1.3.3 Operations on Complex Numbers in Polar Form

Addition and Subtraction of Complex number in Polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The addition and subtraction of two numbers can be computed simply as

$$z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$

and $z_1 - z_2 = r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication of Complex number in Polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The product of two complex numbers can be derived by multiplying them directly and simplifying

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \quad \because\ i^2 = -1$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] \quad \text{(Using trigonometric identities)}$$

Thus, multiplying two complex numbers in polar form involves multiplying their moduli and summing their arguments i.e., $arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$

Example 13: Find the product of $5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$.

Solution: Let $z_1 = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

Here, $r_1 = 5$ and $\theta_1 = \frac{\pi}{6}$, while $r_2 = 4$ and $\theta_2 = \frac{3\pi}{2}$

Substitute this value in the product formula

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$= 5 \times 4\left[\cos\left(\frac{\pi}{6} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + \frac{3\pi}{2}\right)\right] = 20\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Thus, the required product is $20\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

Division of Complex Number in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The formula for division of two complex numbers in polar form can be derived by rationalizing the denominator.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}.\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \quad \begin{pmatrix}\text{Multiply and divide the equation}\\ \text{by conjugate of } \cos\theta_2 + i\sin\theta_2\end{pmatrix}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right] \quad \text{(Using trigonometric identities)}$$

Thus, the modulus of the division of two complex numbers equals the quotient of their moduli, while the arguments of the quotient is the difference between their arguments.

Thus, when dividing two complex numbers, the modulus of the result is the ratio of their moduli, and the argument of the result is the difference between their arguments

i.e., $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Example 14: Divide $\frac{2}{7}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ by $\frac{3}{5}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

Solution: Let $z_1 = \frac{2}{7}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ and $z_2 = \frac{3}{5}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

Here, $r_1 = \frac{2}{7}, \theta_1 = \frac{7\pi}{6}, r_2 = \frac{3}{5}$ $and\,\theta_2 = -\frac{\pi}{2}$.

Substitute value in the quotient formula

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$= \frac{2}{7} \times \frac{5}{3}\left[\cos\left(\frac{7\pi}{6} - \left(-\frac{\pi}{2}\right)\right) + i\sin\left(\frac{7\pi}{6} - \left(-\frac{\pi}{2}\right)\right)\right]$$

$\frac{z_1}{z_2} = \frac{10}{21}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$ is the required polar form of division of two complex number.

Example 15: If $z = x + iy$, then write the equation $|3z - i| = |3\bar{z} + 7|$ in term of x and y.

Solution: Given $|3z - i| = |3\bar{z} + 7|$ ...(i)

$$|3z - i| = |3(x+iy) - i| = |3x + i(3y-1)| = \sqrt{(3x)^2 + (3y-1)^2}$$

$$|3\bar{z} + 7| = |\overline{3x+3iy} + 7| = |3x - 3iy + 7| = |3x + 7 + i(-3y)| = \sqrt{(3x+7)^2 + (-3y)^2}$$

Substitutes these values in (i)

$$\sqrt{(3x)^2 + (3y-1)^2} = \sqrt{(3x+7)^2 + (-3y)^2}$$

Taking square on both sides

$$(3x)^2 + (3y-1)^2 = (3x+7)^2 + (-3y)^2$$

$$9x^2 + 9y^2 - 6y + 1 = 9x^2 + 42x + 49 + 9y^2$$

$$\Rightarrow \quad -6y + 1 = 42x + 49$$

$$\Rightarrow \quad -6y = 42x + 48$$

$$\text{or} \quad y = -7x - 8$$

The equation $y = -7x - 8$ represents a straight line in the complex plane.

Example 16: Show that $(x+2)^2 + y^2 = 8$ if $\arg\left(\dfrac{z+2i}{z-2i}\right) = \dfrac{3\pi}{4}$ for $z = x + iy$.

Solution: $\dfrac{z+2i}{z-2i} = \dfrac{x+iy+2i}{x+iy-2i} = \dfrac{x+i(y+2)}{x+i(y-2)} = \dfrac{x+i(y+2)}{x+i(y-2)} \times \dfrac{x-i(y-2)}{x-i(y-2)}$

$$\Rightarrow \quad \frac{z+2i}{z-2i} = \frac{(x^2+y^2-4)+4ix}{x^2+(y-2)^2} = \frac{x^2+y^2-4}{x^2+(y-2)^2} + i\frac{4x}{x^2+(y-2)^2}$$

As $\quad \arg\left(\dfrac{z+2i}{z-2i}\right) = \dfrac{3\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{4x}{x^2+(y-2)^2}}{\frac{x^2+y^2-4}{x^2+(y-2)^2}}\right) = \frac{3\pi}{4} \quad \Rightarrow \quad \frac{4x}{x^2+y^2-4} = \tan\frac{3\pi}{4} = -1$$

$$\Rightarrow \quad 4x = -1(x^2+y^2-4) \quad \Rightarrow \quad x^2 + 4x + y^2 = 4$$

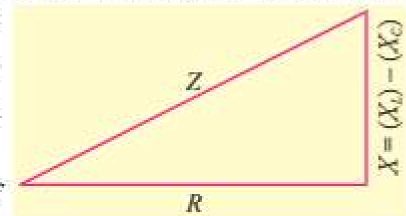
Completing the square for x^2, we have

$$(x+2)^2 + y^2 = 8$$

1.5 Complex Numbers in the Real World (Voltage, Current and Resistance)

Ohm's Law is a fundamental principle in physics that describes the relationship between voltage 'v', current 'I' and resistance 'R' in an electrical circuit. Mathematically Ohm's Law can be expressed by the formula $V = IR$.

when dealing with alternating current (AC) circuits, resistance generalizes to impedance (Z). Resistance in a circuit is due to inductor (X_L) and capacitor (X_C). Their difference is reactance $X = (X_L) - (X_C)$. Geometrically it is shown in the adjacent figure. Here $Z = R + iX$



Then for AC circuits, Ohm's Law in Terms of Impedance is expressed by the formula $V = I \cdot Z$.

Example 17: If the impedance of circuit is $11(\cos 55.35° + i\sin 55.35°)$ ohms at a voltage of $25\left(\cos 30° + i\sin 30°\right)$ V, find the value of current in the circuit.

Solution: Substitute the voltage 25(cos 30° + I sin 30°) and impedance 11(cos 55.35° + I sin 55.35°) into the equation $V = IZ$, where V is voltage, I denote the current and Z is impedance.

$$25\left(\cos 30° + i\sin 30°\right) = I \,.\, 11(\cos 55.35° + i\sin 55.35°)$$

$$\text{or} \qquad I = \frac{25\left(\cos 30° + i\sin 30°\right)}{11(\cos 55.35° + i\sin 55.35°)}$$

$$I = \frac{25}{11}\left[\cos\left(30° - 55.35°\right) + i\sin\left(30° - 55.35°\right)\right]$$

$$I = 2.27\left[\cos\left(-25.35°\right) + i\sin\left(-25.35°\right)\right]$$

Express into rectangular form

$$I = 2.27\left[0.90 + i\left(-0.42\right)\right] = 2.04 - 0.95i$$

Thus, current is $6 - 4.21i$.

Cryptography: It is the science of securing information by transforming readable messages called plaintext into secrete code called ciphertext using mathematical algorithms and encryption keys. It consists of two main processes i.e., encryption to lock message with complex math, and decryption to unlock it with the right key.

Example 18: The word "MATH" is to be encrypted by multiplying a complex number $k = 2 + 3i$ and then decrypted back to its original form using the concept of multiplicative inverse in complex numbers.

Each letter of the alphabet is assigned a numerical value as follows:

$$A = 1, B = 2, C = 3, \ldots, Z = 26$$

Solution: First, we assign each letter in the word "MATH" a complex number with zero imaginary part. The encryption and decryption shown in the table below

Letter	Complex Number (z)	z encrypted = z × k	z decrypted = z **encrypted** / k	Letter
M	$13 + 0i$	$(13 + 0i)(2 + 3i) = 26 + 39i$	$(26 + 39i) / (2 + 3i) = 13 + 0i$	M
A	$1 + 0i$	$(1 + 0i)(2 + 3i) = 2 + 3i$	$(2 + 3i) / (2 + 3i) = 1 + 0i$	A
T	$20 + 0i$	$(20 + 0i)(2 + 3i) = 40 + 60i$	$(40 + 60i) / 2 + 3i = 20 + 0i$	T
H	$8 + 0i$	$(8 + 0i)(2 + 3i) = 16 + 24i$	$16+24i / 2 + 3i = 8 + 0i$	H

EXERCISE 1.5

1. Plot the following points:

   (i) $(2,\ 75^{\circ})$ (ii) $(-3,\ 120^{\circ})$ (iii) $\left(2,\ \frac{\pi}{6}\right)$ (iv) $\left(5,\ \frac{5\pi}{6}\right)$

   (v) $\left(-\frac{5}{2},\ \frac{\pi}{3}\right)$ (vi) $\left(-3,\ -\frac{2\pi}{3}\right)$ (iii) $\left(-\frac{9}{2},\ -\frac{19\pi}{12}\right)$ (iv) $\left(-\frac{5}{2},\ \frac{5\pi}{12}\right)$

2. Express the following complex numbers in polar form :

   (i) $4+3i$ (ii) $1+i$ (iii) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (iv) $-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$

   (v) $-\frac{5}{7}+\frac{\sqrt{8}}{7}i$ (vi) $\frac{1}{2}+\frac{\sqrt{5}}{2}i$ (vii) $-\frac{2}{3}-\frac{\sqrt{7}}{3}i$

3. Convert each of the complex number z in the rectangular form $x+iy$:

   (i) $4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$ (ii) $\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

   (iii) $|z|=7,\ arg(z)=\frac{23\pi}{12}$ (iv) $|z|=11,\ arg(z)=-\frac{11\pi}{12}$

   (v) $|z|=\frac{10}{3},\ arg(z)=-\frac{17\pi}{12}$ (vi) $2\cos(-33)+i\,2\sin(-33)$

   (vii) $|z|=12,\ arg(z)=\pi$

4. If $z_1=9\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$ and $z_2=5\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ then find

   (i) z_1+z_2 (ii) z_1-z_2 (iii) $z_1\cdot z_2$ (iv) $\frac{z_1}{z_2}$

5. If $z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$ and $z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$ then find the following and express the result into $x + iy$ form

(i) $z_1 + z_2$ (ii) $z_1 - z_2$ (iii) $z_1 \cdot z_2$ (iv) $\frac{z_1}{z_2}$

6. Divide $z_1 = 6(\cos 150° + i\sin 150°)$ by $z_2 = 3(\cos 30° + i\sin 30°)$ and express in $x + iy$ form.

7. Multiply $z_1 = 2(\cos 60° + i\sin 60°)$ and $z_2 = 5(\cos 90° + i\sin 90°)$ and express in $x + iy$ form.

8. Find the modulus and argument of $z = -2 - 2i$.

9. Write the equation $\arg\left(\bar{z} - 2 + i\right) = \frac{2\pi}{3}$ in cartesian form, if $z = x + iy$.

10. If $z = x + iy$ and $\arg\left(\frac{\bar{z} - 1 + 2i}{\bar{z} + 1 - 2i}\right) = \frac{9\pi}{4}$, show that $x^2 + y^2 - 4x + 2y - 5 = 0$.

11. If $z = x + iy$ and $\arg(z - 2 - 3i) - \arg(z + 2 + 3i) = 2\pi$, show that $2y = 3x$.

12. Solve the equation $|z - 2i| = |\bar{z} + 2|$ for $z = x + iy$.

13. For $z = x + iy$, solve the equation $|5z + 4 + i| = |5\bar{z} - 3 + 2i|$.

14. Determine the set of points $z = x + iy$ that satisfy the equation $|3\bar{z} - 2 + i| = |3z + i|$.

15. An AC source supplies a voltage of $V = 120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ volts to a circuit with impedance $Z = \frac{1 + i\sqrt{3}}{2}$ ohms. Calculate the current in polar form.

16. An AC circuit has an impedance of $Z = 3 - 6i$ ohms and is connected to a voltage source of $V = 90 + 30i$ volts. Find the current in both rectangular and polar form.

17. Encrypt the word "CODE" by multiplying the complex encryption key $k = 2 - i$. Then decrypt it back to the original word.

18. Consider the complex encryption key $k = 3 - 3i$. Encrypt the word "QUIZ", and then recover the original word using the inverse of the key.

19. Encrypt the word "CLASS" by adding the complex encryption key $k = -3 + 4i$. Then decrypt it back to the original word.

Unit 2

INTRODUCTION

Functions are fundamental in mathematics, describing relationships between inputs and outputs through a rule of correspondence. Understanding key concepts such as domain, co-domain and range is essential for analyzing different types of functions, including one-to-one, onto and bijective functions. Graphical representation helps in identifying intersecting points, such as where a linear function meets the coordinate axes, where two linear functions intersect or where a linear and a quadratic function cross. These intersections provide valuable insights into solving equations visually. Additionally, exploring square root and cube root function graphs allows for a deeper understanding of their unique properties and behaviour. This unit will enhance problem-solving skills by combining algebraic and graphical approaches to functions.

2.1 Concept of Function

The term function was recognized by a German Mathematician Leibniz (1646-1716) to describe the dependence of one quantity on another. The following examples illustrate how this term is used:

(i) The area "A" of a square depends on one of its sides "x" by the formula $A = x^2$, so we say that A is a function of x.

(ii) The volume "V" of a sphere depends on its radius "r" by the formula $V = \frac{4}{3}\pi r^3$, so we say that V is a function of r.

A **function** is a rule or correspondence, relating two sets in such a way that each element in the first set corresponds to one and only one element in the second set.

Thus in, (i) above, a square of a given side has only one area and in, (ii) above, a sphere of a given radius has only one volume.

Now we have a formal definition:

2.1.1 Definition (Function, Domain, Codomain, Range)

A **function** f from a set X to a set Y is a rule or a correspondence that assigns to each element x in X a unique element y in Y. The set X is called the **domain** of f.

The set of corresponding elements y in Y is called the **range** of f. While the **codomain** of a function is the set Y in which function's output values (range) lie.

Unless stated to the contrary, we shall assume hereafter that the set X and Y consist of real numbers.

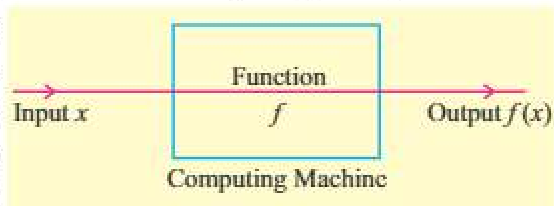
Note: Co-domain is the set of all possible outputs but the range is the actual set of outputs produced by the function under the given domain that is range set is always a subset of co-domain.

2.1.2 Notation and Value of a Function

If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that "y is a function of x".

Swiss mathematician Euler (1707 – 1783) invented a symbolic way to write the statement "y is a function of x" as $y = f(x)$, which is read as "y is equal to f of x".

A function can be thought as a computing machine f that takes an input x, operates on it in some way and produces exactly one output $f(x)$. This output $f(x)$ is called the value of f at x or image of x under f.



the output $f(x)$ is denoted by a single letter, say y and we write $y = f(x)$.

The variable x is called the **independent variable** of f and the variable y is called the **dependent variable** of f. For now onward we shall only consider the function in which the variables are real numbers and we say that f is a **real valued function of real numbers**.

Example 1: Given $f(x) = x^3 - 2x^2 + 4x - 1$, find: (i) $f(0)$ (ii) $f(1)$ (iii) $f(-2)$ (iv) $f(1 + x)$ (v) $f\left(\frac{1}{x}\right), x \neq 0$

Solution: $f(x) = x^3 - 2x^2 + 4x - 1$

(i) $f(0) = 0 - 0 + 0 - 1 = -1$

(ii) $f(1) = (1)^3 - 2(1)^2 + 4(1) - 1 = 1 - 2 + 4 - 1 = 2$

(iii) $f(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 1 = -8 - 8 - 8 - 1 = -25$

(iv) $f(1 + x) = (1 + x)^3 - 2(1 + x)^2 + 4(1 + x) - 1$

$= 1 + 3x + 3x^2 + x^3 - 2 - 4x - 2x^2 + 4 + 4x - 1$

$= x^3 + x^2 + 3x + 2$

(iv) $f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) - 1 = \frac{1}{x^3} - \frac{2}{x^2} + \frac{4}{x} - 1, \quad x \neq 0$

Example 2: Find the domain and range of $f(x) = x^2$.

Solution: For every real number x, $f(x) = x^2$ is a non-negative real number. So,
Domain f = set of all real numbers ; Range f = set of all non-negative real numbers.

Example 3: Find the domain and range of $f(x)=\dfrac{x}{x^2-4}$.

Solution: At $x=2$ and $x=-2$, $f(x)=\dfrac{x}{x^2-4}$ is not defined. So,

Domain f = set of all real numbers except –2 and 2 or $R-\{-2,2\}$

Let $y=\dfrac{x}{x^2-4}\Rightarrow y(x^2-4)=x\Rightarrow x^2y-4y=x$

$$x^2y-x-4y=0$$

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(y)(-4y)}}{2y}$$

$$x=\frac{1\pm\sqrt{1+16y^2}}{2y},\ y\neq 0$$

x is defined as $\forall\ \ y\neq 0$

For $y=0$, we have $0=\dfrac{x}{x^2-4}\Rightarrow x=0$

$f(0)=0$

So, range f = set of all real numbers or $(-\infty,\infty)$

Remember!

There are two types of intervals known as open interval and closed interval. In an open interval (a, b), the endpoints are not included. In a closed interval [a, b], the endpoints are included.

Example 4: Find the domain and range of $f(x)=\sqrt{x^2-9}$.

Solution: $\sqrt{x^2-9}\geq 0\Rightarrow x^2-9\geq 0$ …(i)

Let $x^2-9=0\Rightarrow x=\pm 3$

Critical points divide the number line into three regions:

Put $x=-4$ in (i), $16-9\geq 0$ (True)

Put $x=0$ in (i), $0-9\geq 0$ (False)

Put $x=4$ in (i), $16-9\geq 0$ (True)

So, domain $f=(-\infty,-3]\cup[3,\infty)$

The smallest value of x^2-9 is 0 (when $x=\pm 3$).

$\Rightarrow\ \ y=\sqrt{0}=0$

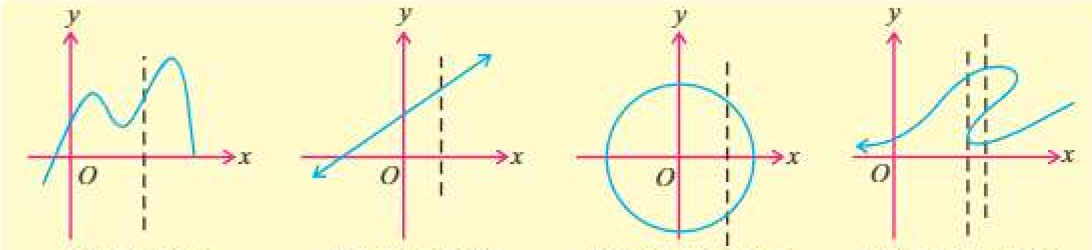
As $|x|$ increases beyond 3, x^2-9 grows to $+\infty$, so y grows to $+\infty$.

So, range $f=[0,\infty)$

2.1.3 Vertical Line Test

The vertical line test is a method used to determine whether a graph represents a function. A graph represents a function if and only if no vertical line intersects the graph more than once. If any vertical line passes through the graph more than once, it is not a function.

Explanation is given in the figure.



(a) a function (b) a function (c) not a function (d) not a function

2.1.4 Types of Function

(i) One-to-One (Injective) Function

A function f is one-to-one if different inputs produce different outputs, i.e., if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. This means that no two different elements of the domain map to the same element of the co-domain.

For example, $f(x) = 5x + 7$ is one-to-one because if $5x_1 + 7 = 5x_2 + 7$ implies $x_1 = x_2$.

(ii) Onto (Surjective) Function

A function $f : X \to Y$ is called onto (or surjective) function if every element in the co-domain Y has at least one pre-image in the domain X. In other words, for every y in Y, there exists an x in X such that $f(x) = y$.

For example, $f(x) = 2x + 3$, where the domain and co-domain are both real numbers.

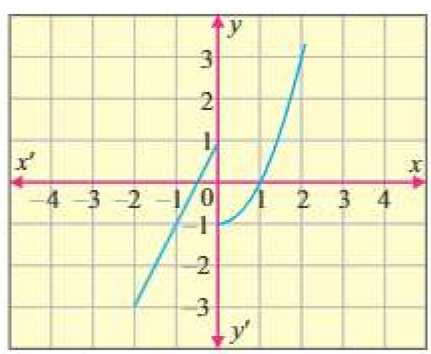
Here $y = 2x + 3 \Rightarrow x = \dfrac{y-3}{2}$. Here for each y in R, there exists $\dfrac{y-3}{2}$ in R such that $f\left(\dfrac{y-3}{2}\right) = y$. Hence f is an onto function.

(iii) Bijective Function

A function $f : X \to Y$ is called bijective if it is both one-to-one and onto.

Piecewise Function

A piecewise function is a function that is defined by different expressions (or "pieces") over different intervals of its domain. Each piece applies to a specific part of the domain.



For example, $f(x) = \begin{cases} 2x+1 & \text{if } x < 0 \\ x^2 - 1 & \text{if } x \geq 0 \end{cases}$

For $x < 0$, the function behaves like $2x + 1$ and for $x \geq 0$, it behaves like $x^2 - 1$

Example 5: Show that the function $f(x) = x+1$, where the domain and co-domain are all real numbers, is bijective.

Solution: A function is bijective if it is both one-to-one and onto.

A function is one-to-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $f(x) = x+1$

Suppose $f(x_1) = f(x_2)$

$$x_1 + 1 = x_2 + 1$$

$$\Rightarrow \quad x_1 = x_2$$

So, the given function is one-to-one.

It is also onto because for every real number y, there is a real number x (specifically $x = y-1$) such that $f(y-1) = y - 1+1 = y$. Therefore, $f(x)$ is bijective.

Example 6: Show that the function $f(x) = x^2 - 2$, where the domain and co-domain are all real numbers, is neither one-to-one nor onto.

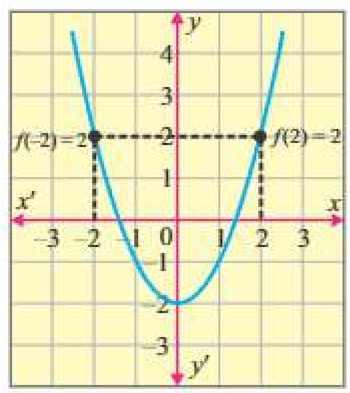
Solution: As $f(x_1) = f(x_2) \Rightarrow x_1^2 - 2 = x_2^2 - 2 \Rightarrow x_1^2 = x_2^2$

Taking square root, we get $x_1 = x_2$ or $x_1 = -x_2$

This does not imply that $x_1 = x_2$, for example

$x_1 = 2, x_2 = -2 \Rightarrow x_1 \neq x_2$ and $f(2) = 2 = f(-2)$.

Thus, f is not one-to-one.



Also, the element –2 in the co-domain R is the smallest value that $f(x) = x^2 - 2$ can attain, and it is only achieved when $x = 0$. However, any number less than –2 (in co-domain R) is not the image of any real number x in domain R. For example, $f(x) = -3 \Rightarrow x^2 - 2 = -3$ has no real root.

EXERCISE 2.1

1. Given that: (a) $f(x) = x^2 - 1$ (b) $f(x) = \sqrt{2x+3}$ Find:

   (i) $f(-3)$ (ii) $f(0)$ (iii) $f(x-2)$ (iv) $f(x^2+3)$

2. Find $\dfrac{f(a+h) - f(a)}{h}$ and simplify where,

   (i) $f(x) = 4x + 7$ (ii) $f(x) = \sin x$

   (iii) $f(x) = x^3 + x^2 - 1$ (iv) $f(x) = \tan x$

3. Express the following:
   (a) The area A of a square as a function of its perimeter P.
   (b) The circumference C of a circle as a function of its area A.
   (c) The surface area S of a cube as a function of its volume V.
4. Find the domain and the range of the function g defined below:
   (i) $g(x) = 5 - x$ (ii) $g(x) = \sqrt{x+2}$
   (iv) $g(x) = \begin{cases} 6x+7, & x \leq -2 \\ 4-3x, & x > -2 \end{cases}$ (v) $g(x) = |x - 5|$
5. Given $f(x) = x^3 - ax^2 + bx + 1$. If $f(2) = -3$ and $f(-1) = 0$. Find the values of a and b.
6. Find the domain and range of $g(x) = \dfrac{x+2}{3-x}$
7. A stone falls from a height of 60m on the ground, the height h after x seconds is approximately given by $h(x) = 40 - 10x^2$.
   (i) What is the height of stone when:
      (a) $x = 1$ sec ? (b) $x = 1.5$ sec ? (c) $x = 1.7$ sec ?
      (ii) When does the stone strike the ground?
8. Consider the function $f(x) = 3x - 5$.
   (i) Determine the domain and range of $f(x)$.
   (ii) Is the function f one-to-one? Justify your answer.
   (iii) Is the function f onto if the co-domain is all real numbers? Explain.
9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \dfrac{2x-3}{x+1}$
   (i) Find the domain and range of $f(x)$. (ii) Determine whether $f(x)$ is onto.
   (iii) Prove that $f(x)$ is one-to-one.
10. Consider the function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = e^{-x}$. Show that $f(x)$ is a bijective.
11. Let g: $\mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = x^3 - 3x$. Determine if $g(x)$ is injective and/or surjective.

2.2 Finding the Intersecting Point(s) Graphically

The point of intersection is a point where two or more graphs meet on the coordinate plane. This point represents the solution(s) to the equations of the given functions.

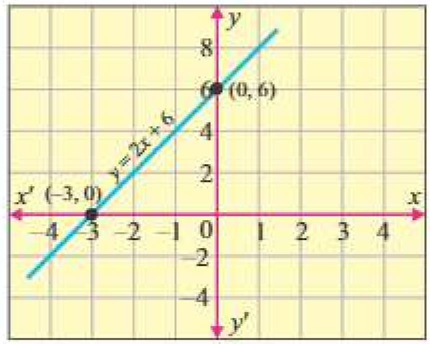
2.2.1 Intersection of a Linear Function and Coordinate Axes

As we know that linear function is a function in which the highest power of the variable is one. While the coordinate axes refers to x-axis and y-axis in the Cartesian coordinate system.

Example 7: Find the points of intersection of a linear function $y = 2x + 6$ and coordinate axes.

Solution: Table values and the graph of $y = 2x + 6$ is given below:

x	$y = 2x + 6$
–1	4
0	6
1	8



Hence, from the above graph, the points (–3, 0) and $(0, 6)$ are the points of intersections of $y = 2x + 6$ and coordinate axes.

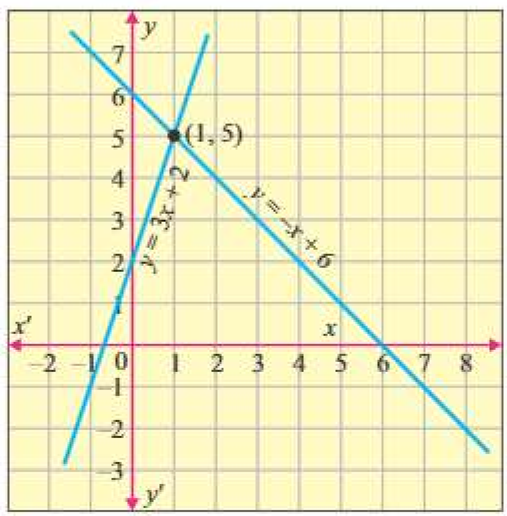
2.2.2 Intersection of Two Linear Functions

The point of intersection of two linear functions is the point where their graphs cross each other. This means the two functions have the same x and y values at that point.

Example 8: Find the point of intersection of $y = 3x + 2$ and $y = -x + 6$.

Solution: Table of different values of x and y is given below:

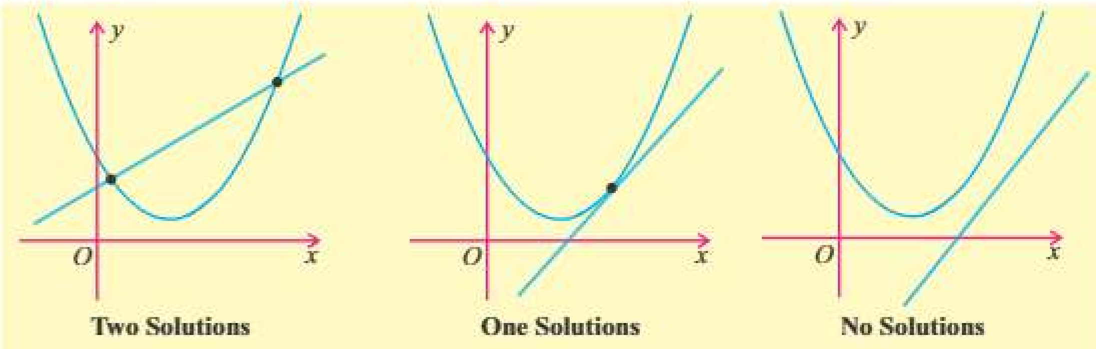
x	$y = 3x + 2$	$y = -x + 6$
–1	–1	7
0	2	6
1	5	5



By plotting the above points, we see that (1, 5) is the point of intersection of both the straight lines as shown in figure.

2.2.3 Intersection of a Linear Function and a Quadratic Function

A line and a parabola can either intersect at two points, one point or not intersect at all. If there are two solutions, the system has two points of intersection. A single solution indicates that there is only one intersection point, suggesting that the line may be tangent to the parabola. If no solution exists, it means the line and the parabola do not intersect.



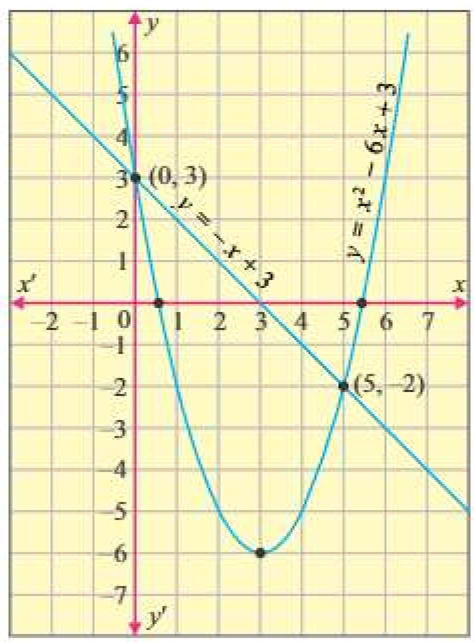
Example 9: Solve the linear function $y=-x+3$ and quadratic function $y=x^2-6x+3$ graphically.

Solution: Clearly (3, 0) and (0, 3) are the x-intercept and y-intercept respectively of $y=-x+3$.

$$y=x^2-6x+3 \qquad \ldots\text{(i)}$$

Put $x=0$ in (i), so (0, 3) is the y-intercept.

Put $y=0$ in (i), we have



$$0=x^2-6x+3$$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(3)}}{2(1)}$$

$$x=\frac{6\pm\sqrt{36-12}}{2}$$

$$x=\frac{6\pm\sqrt{24}}{2}$$

$$x=\frac{6\pm2\sqrt{6}}{2}$$

$$x=3\pm\sqrt{6}$$

$$x=3-\sqrt{6},\ 3+\sqrt{6}$$

$$x=0.6,\ 5.4$$

So (0.6, 0) and (5.4, 0) are the x-intercepts.

Now we find vertex (h, k) of the parabola

$$h = -\frac{b}{2a} = -\frac{-6}{2(1)} = 3$$

$$k = (3)^2 - 6(3) + 3 = -6$$

So, the vertex is $(3, -6)$

Hence $(0, 3)$ and $(5, -2)$ are the solutions (points of intersection) of the given functions.

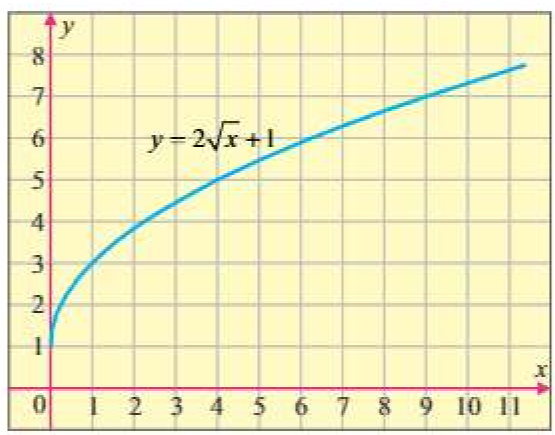
2.3 Graph of the Square Root Function

Example 10: Graph the square root function $y = 2\sqrt{x} + 1$

Solution: Clearly the domain of $y = 2\sqrt{x} + 1$ is $x \geq 0$, as the square root of a negative number is not a real number. The range of $y = 2\sqrt{x} + 1$ is $y \geq 0$, as the square root of a non-negative number is also non-negative.

Table values and the graph of the function are given below:

x	$y = 2\sqrt{x} + 1$
0	1
1	3
2	3.8
3	4.5
4	5
5	5.5
6	5.9
7	6.3
8	6.7
9	7
10	7.3



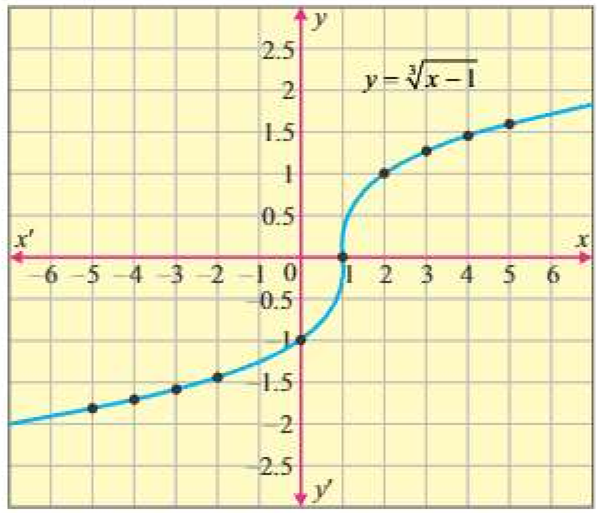
2.4 Graph of the Cube Root Function

Example 11: Graph the cube root function $y = \sqrt[3]{x - 1}$

Solution: As we know that cube root function is defined for all real numbers because the cube root of any number (positive, negative or zero) is always real. Therefore, the domain of the given cube root function is all real numbers. The range of the given function is also the set of real numbers.

Table values and the graph of the function are given below:

x	$y=\sqrt[3]{x-1}$
–5	–1.8
–4	–1.7
–3	–1.6
–2	–1.4
–1	–1.3
0	–1
1	0
2	1
3	1.3
4	1.4
5	1.6



2.5 Real Life Applications

Growth and Decay in Finance (Predicting Long-Term Stock Prices)

When something increases in quantity or size over time, it is called **growth**. For example, money in a bank account earning interest (it grows larger), a population of rabbits is increasing over months.

When something decreases in quantity or size over time, it is called decay. For example, a radioactive substance is losing its strength over years, a cup of hot coffee is cooling down over time.

Example 12: The value of a stock follows the exponential growth model $P(t) = P_0e^{rt}$, where P_0 is the initial stock price, r is the growth rate per year and t is the time in years. Suppose a stock is currently valued at Rs. 5,000, and it is expected to grow at a rate of 5% per year.

(i) Find the value of the stock after 10 years.

(ii) After how many years will the stock double in value?

Solution: (i) The formula for the exponential growth is:

$$P(t) = P_0e^{rt}$$

Given $P_0 = 5{,}000$, $r = 0.05$ (5% growth rate), and $t = 10$ years.

$$P(10) = 5.000\ e^{0.05\times10} = 5{,}000\ e^{0.5}$$

Using $e^{0.5} \approx 1.6487$

$$P(10) = 5{,}000 \times 1.6487 = 8244$$

So, the value of the stock after 10 years is approximately Rs. 8244.

(ii) We want to find t when the stock doubles, i.e., when $P(t) = 2P_0$. Using the equation:

$$2P_0 = P_0 e^{rt}$$

Dividing both sides by P_0, we have $2 = e^{rt}$

Taking the natural logarithm on both sides: $\ln 2 = rt$

and $t = \ln 2 / r = 0.69310.05 = 13.86$

So, the stock will double in value in approximately 13.86 years.

Example 13: The concentration of a pollutant in a lake, in parts per million (ppm), decays over time according to the function

$$C(t) = \frac{100}{\sqrt{t+1}}$$

where t is the time in days since the pollutant was introduced.

(i) What is the concentration of the pollutant after 4 days?

(ii) After how many days will the concentration drop below 10 ppm?

Solution: (i) The pollutant concentration function is $C(t) = \frac{100}{\sqrt{t+1}}$, where t is the time in days.

Concentration after 4 days:

$$C(4) = \frac{100}{\sqrt{4+1}} = \frac{100}{\sqrt{5}} \approx 44.72\,\text{ppm}$$

The concentration after 4 days is about 44.72 ppm.

(ii) When will the concentration drop below 10 ppm? Set $C(t) = 10$:

$$10 = \frac{100}{\sqrt{t+1}} \Rightarrow \sqrt{t+1} = 10 \Rightarrow t+1 = 100 \Rightarrow t = 99$$

After 99 days, the concentration will drop below 10 ppm.

EXERCISE 2.2

1. Find the point of intersection of the coordinate axes and the following linear functions graphically:

(i) $y = -5x + 10$ (ii) $y = 2x - 1$

(iii) $y = \frac{1}{2}x - 3$ (iv) $y = 3x + \frac{3}{2}$

2. Find the point(s) of intersection of the following functions graphically:

(i) $f(x) = 2x + 5$, $g(x) = -x + 5$

(ii) $f(x) = 3x - 2$, $g(x) = 10 - x$

(iii) $f(x) = 2x - 4$, $g(x) = 3x - 1$

(iv) $f(x) = -3x - 4$, $g(x) = \frac{1}{2}x + 3$

(v) $f(x) = x - 1$, $g(x) = x^2 - 4x + 3$

(vi) $f(x) = 3x + 4$, $g(x) = x^2 + 2x - 8$

3. Graph the following functions:

(i) $y = \sqrt{3x}$ (ii) $y = \sqrt{x} + 5$ (iii) $y = -\frac{1}{2}\sqrt{x}$

(iv) $y = -\sqrt{x+1} + 2$ (v) $y = \sqrt[3]{2x+1}$ (vi) $y = 2\sqrt[3]{x} - 3$

(vii) $y = \sqrt[3]{x^2 + x - 2}$

4. A building's height over time is modeled by $H(t) = 100 + 20t$ which is in metres and t is the time in months. The height of a growing tree nearby is given by $T(t) = 50 + 10t + t^2$.

(i) At what time will the building and tree have the same height?

(ii) What will that height be?

Sketch the graphs of both functions and determine the time when the tree will overtake the height of the building.

5. A radioactive substance has a half-life of 2 years. If the initial quantity is 200 grams and the exponential decay function is $Q(t) = Q\left(\frac{1}{2}\right)^{\frac{t}{h}}$, then find the remaining quantity after 6 years graphically?

Unit 3 Theory of Quadratic Functions

INTRODUCTION

This unit explores methods to find the maximum and minimum values of quadratic functions using completing the square and graphical analysis. It also covers the inverse of quadratic functions, determining their domain and range. Additionally, students will learn to solve absolute value quadratic equations and inequalities, as well as equations of rational, radical and exponential forms that can be reduced to quadratic equations. Finally, the unit demonstrates the practical applications of quadratic equations and inequalities in solving real-world problems, providing a strong foundation for problem-solving and analysis.

3.1 Quadratic Function

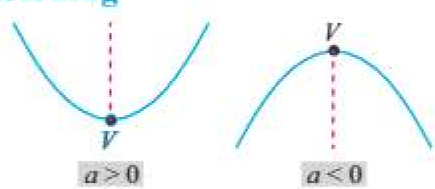
A quadratic function is a polynomial function of degree two. It is typically expressed in the standard form:

$$f(x) = ax^2 + bx + c$$

where a, b and c are real numbers, and $a \neq 0$.

3.1.1 Analyzing Quadratic Function by Sketching

As we know shape of the graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola. The parabola opens upward or downward, depending on the sign of the leading coefficient a, as shown in the given figure.



The tip of the parabola, labeled as **V** in the diagrams above, is known as the vertex having coordinates (h, k). The vertical line passing through the vertex serves as the axis of symmetry for the parabola. The vertex represents a turning point, where the graph changes direction.

- If $a > 0$, then the vertex is a minimum point.
- If $a < 0$, then the vertex is a maximum point.

For sketching the quadratic function, we need to find the x-intercept, y-intercept and the vertex. For analyzing the sketch of quadratic function, we find whether the vertex is a minimum or a maximum point and indicate the intervals where the function is increasing or decreasing.

Example 1: Sketch and analyze $y = -x^2 - 2x + 3$.

Solution: $y = -x^2 - 2x + 3$

The y-intercept is $y = -(0)^2 - 2(0) + 3 = 3$

The x-intercepts are found by solving the equation:

$$-x^2 - 2x + 3 = 0 \quad \text{or} \quad x^2 + 2x - 3 = 0$$

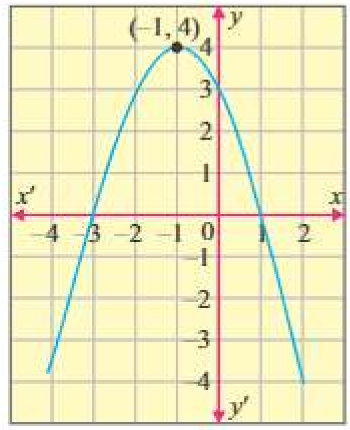
$$x^2 + 3x - x - 3 = 0$$

$$x(x+3) - 1(x+3) = 0$$

$$(x+3)(x-1) = 0$$

$$x + 3 = 0,\ x - 1 = 0$$

$$x = -3,\ x = 1$$



Now, we find the vertex

$$h = \frac{-b}{2a} = -\frac{(-2)}{2(-1)} = -1$$

$$k = -(-1)^2 - 2(-1) + 3 = -1 + 2 + 3 = 4$$

So, the vertex $(-1, 4)$ is a maximum point. The function y is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$

3.1.2 Finding Maximum and Minimum Values of Quadratic Functions by Completing Square

Completing the square is a technique used to rewrite a quadratic function in the following vertex form:

$$f(x) = a(x-h)^2 + k$$

Where vertex $= (h, k)$, $h = -\dfrac{b}{2a}$ and $k = c - \dfrac{b^2}{4a}$

- If $a > 0$, the minimum value of $f(x)$ at $x = h$ is k.
- If $a < 0$, the maximum value of $f(x)$ at $x = h$ is k.

Example 2: Find the maximum or minimum value of $f(x) = -2x^2 + 4x + 3$ by completing square.

Solution:

$$f(x) = -2(x^2 - 2x) + 3$$

$$f(x) = -2(x^2 - 2x + 1 - 1) + 3$$

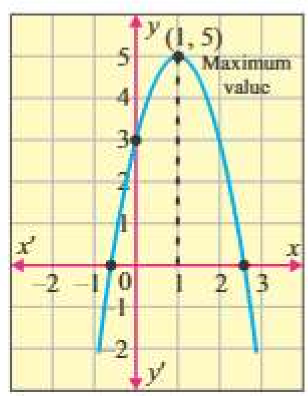
$$f(x) = -2[(x-1)^2 - 1] + 3$$

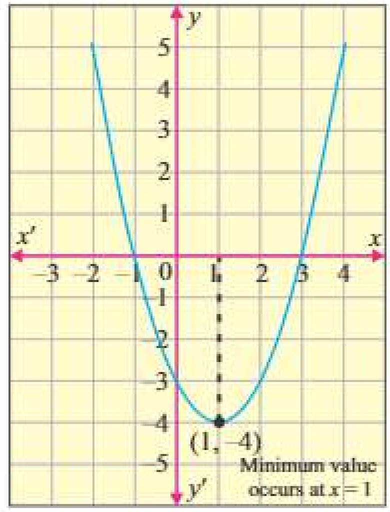
$$f(x) = -2(x-1)^2 + 2 + 3$$

$$f(x) = -2(x-1)^2 + 5$$

Here $a = -2 < 0$

Therefore, the maximum value is 5, which occurs when $x = 1$.





Example 3: Find the maximum or minimum value of $f(x) = x^2 - 2x - 3$.

Solution: Given that $f(x) = x^2 - 2x - 3$

Here $a = 1$, $b = -2$, $c = -3$

$$h = -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$$

and $$k = c - \frac{b^2}{4a} = -3 - \frac{(-2)^2}{4(1)} = -4$$

Here $a = 1 > 0$

Therefore, the minimum value of $f(x)$ at $x = 1$ is -4.

3.2 Inverse of Quadratic Function

Quadratic functions are typically not one-to-one over their entire domain. To find an inverse for a quadratic function, we must restrict its domain to a portion where it is one-to-one. Commonly, we restrict the domain to either $x \geq h$ (where h is the x-coordinate of the vertex) or $x \leq h$.

Example 4: Find the inverse of $f(x) = x^2 + 4x + 3, x \geq -2$. Also find its domain and range.

Solution: $f(x) = x^2 + 4x + 3 \quad , \quad x \geq -2$

$$y = x^2 + 4x + 3$$

$$x = y^2 + 4y + 3$$

$y^2 + 4y + 3 - x = 0$ (Interchange x and y)

$$y = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(3-x)}}{2(1)}$$ (Using the quadratic formula)

$$y = \frac{-4 \pm \sqrt{16 - 12 + 4x}}{2}$$

$$y = \frac{-4 \pm \sqrt{4 + 4x}}{2}$$

$$y = \frac{-4 \pm 2\sqrt{1 + x}}{2}$$

$f^{-1}(x) = -2 \pm \sqrt{1 + x}$ (Replace y with $f^{-1}(x)$)

The above inverse function has both a positive and a negative component. To determine which is the inverse, we find domain and range of the given function.

Domain $f = [-2, \infty)$

To find range, we proceed as

Since $x \geq -2$

$$x^2 \geq +4$$
$$4x \geq -8$$
$$x^2 + 4x \geq -4$$
$$x^2 + 4x + 3 \geq -4 + 3$$

$\Rightarrow \quad f(x) \geq -1$

As $\quad f(x) = x^2 + 4x + 3$

$\Rightarrow \quad f(x) = (x+2)^2 - 1$

Therefore, minimum value of $f(x)$ is $-1$and hence

Range $f = [-1, \infty)$

Domain $f^{-1} = [-1, \infty)$, Range $f^{-1} = [-2, \infty)$

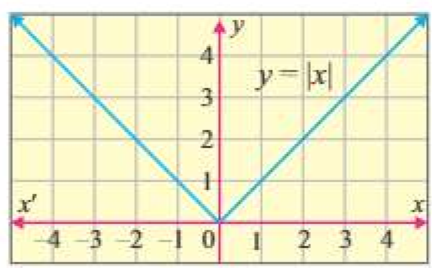
Now, we substitute any value of x that falls within the domain. We choose the value $x = 0$.

$$f^{-1}(0) = -2 + \sqrt{1+0} = -1$$
$$f^{-1}(0) = +2 - \sqrt{1-0} = -3$$

We notice only -1 lies in the range of f. Therefore, we discard negative component.

Hence $f^{-1}(x) = -2 + \sqrt{1+x}$

3.3 Absolute Value



The absolute value of x, is defined as

$$|x| = \begin{cases} x \ , & x \geq 0 \\ -x \ , & x < 0 \end{cases}$$

3.3.1 Absolute Value Quadratic Equations

To solve the absolute value quadratic equations, all answers must be substituted back into the original equation to verify whether they are valid or not. Sometimes, "extraneous" solutions may appear which are not valid and must be eliminated from the final answer.

Example 5: Solve $|x^2 - 4| = 5$

Solution: $|x^2 - 4| = 5$

$$x^2 - 4 = \pm 5$$
$$x^2 - 4 = 5 \quad \text{or} \quad x^2 - 4 = -5$$
$$x^2 = 9 \quad \text{or} \quad x^2 = -1$$
$$x = \pm 3 \quad \text{or} \quad x = \pm\sqrt{-1} = \text{imaginary}$$

Check:

For $x = 3$	For $x = -3$
$\lvert 3^2 - 4\rvert = 5$	$\lvert(-3^2) - 4\rvert = 5$
$\lvert 5\rvert = 5$	$\lvert 5\rvert = 5$
$5 = 5$	$5 = 5$

Hence solution set = $\{\pm 3\}$

3.3.2 Absolute Value Quadratic Inequalities

Absolute value quadratic inequalities are inequalities that involve a quadratic expression within absolute value bars. They are generally of the following form:

$|ax^2 + bx + c| < d$, $|ax^2 + bx + c| > d$, $|ax^2 + bx + c| \le d$, $|ax^2 + bx + c| \ge d$

Example 6: Solve $\left|x^2 - 6x - 4\right| < 3$

Solution: $|x^2 - 6x - 4| < 3$

$-3 < x^2 - 6x - 4 < 3$

$-3 < x^2 - 6x - 4$ or $x^2 - 6x - 4 < 3$

$x^2 - 6x - 4 + 3 > 0$ or $x^2 - 6x - 4 - 3 < 0$

$x^2 - 6x - 1 > 0$ (i) , $x^2 - 6x - 7 < 0$ (ii)

Here we solve $x^2 - 6x - 1 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 + 4}}{2}$$

$$x = \frac{6 \pm \sqrt{40}}{2}$$

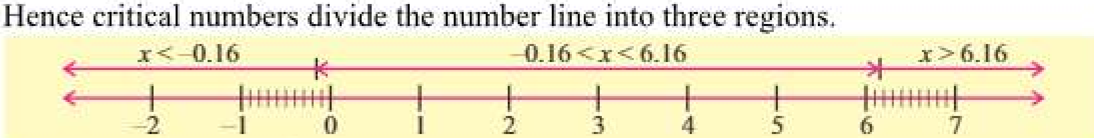
$$x = \frac{6 \pm 2\sqrt{10}}{2}$$

$$x = 3 \pm \sqrt{10}$$

$$x = 3 - \sqrt{10} \quad , \quad 3 + \sqrt{10}$$

$$x = -0.16 \quad , \quad 6.16$$

Hence critical numbers divide the number line into three regions.



Test $x = -1$ in (i), we have

$(-1)^2 - 6(-1) - 1 > 0 \Rightarrow +6 > 0$ (True)

Test $x = 0$ in (i), we have

$$(0)^2 - 6(0) - 1 > 0 \quad \Rightarrow \quad -1 > 0 \quad \text{(False)}$$

Test $x = 7$ in (i), we have

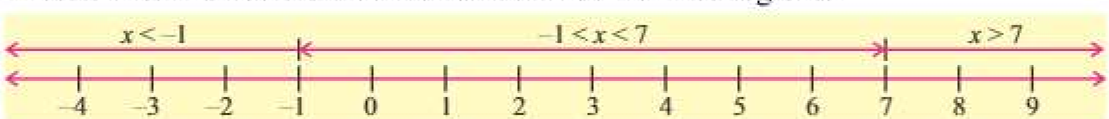
$$(7)^2 - 6(7) - 1 > 0 \quad \Rightarrow \quad 6 > 0 \quad \text{(True)}$$

Solution set is $(-\infty, -0.16) \cup (6.16, \infty)$

Now, we take (ii) and solve

$$x^2 - 6x - 7 = 0$$
$$x^2 + x - 7x - 7 = 0$$
$$x(x+1) - 7(x+1) = 0$$
$$(x+1)(x-7) = 0$$
$$x + 1 = 0 \quad , \quad x - 7 = 0$$
$$x = -1 \quad , \quad x = 7$$

These critical numbers divide the number line into three regions.



Test $x = -2$, $x = 0$ and $x = 10$ in (ii), we have

$$(-2)^2 - 6(-2) - 7 < 0 \quad \Rightarrow \quad 9 < 0 \quad \text{(False)}$$
$$(0)^2 - 6(0) - 7 < 0 \quad \Rightarrow \quad -7 < 0 \quad \text{(True)}$$
$$(10)^2 - 6(10) - 7 < 0 \quad \Rightarrow \quad 33 < 0 \quad \text{(False)}$$

Solution set is $(-1, 7)$

Hence the solution set of the given absolute value quadratic inequality is

$(-\infty, -0.16) \cup (6.16, \infty)\} \cap \{(-1, 7) = (-1, -0.16) \cup (6.16, 7)$

EXERCISE 3.1

1. Find the maximum or minimum value of the following quadratic functions by completing square:

   (i) $f(x) = x^2 + 6x + 13$ (ii) $f(x) = x^2 + 4x$

   (iii) $f(x) = -x^2 + 8x + 13$ (iv) $f(x) = -x^2 - 3x - 5$

   (v) $f(x) = 3x^2 + 6x - 13$ (vi) $f(x) = -2x^2 - x + 21$

2. Find the maximum or minimum point by sketching the following quadratic functions. Also find their domain and range:

   (i) $f(x) = x^2 - 4x$ (ii) $f(x) = x^2 - 5x + 6$

   (iii) $f(x) = -x^2 + 2x - 8$ (iv) $f(x) = x^2 - 4x + 4$

   (v) $f(x) = x^2 + 2x - 8.3$ (vi) $f(x) = 6 - x - x^2$

3. Find the inverse of the following quadratic functions. Also find their domain and range:

(i) $f(x) = x^2 - 3, \quad x \le 0$ (ii) $f(x) = x^2 + 6x + 4, \quad x < -3$

(iii) $f(x) = 2x^2 - 8x + 11, \quad x \ge 2$ (iv) $f(x) = 3x^2 - 2x + 6, \quad x \ge 5$

(v) $f(x) = 2(x-3)^2 + 1, \quad x \ge 3$ (vi) $f(x) = -3(x+4)^2 - 5, \quad x < -4$

4. Solve the following absolute value quadratic equations and inequalities:

(i) $|x^2 + 1| = 5$ (ii) $|x^2 + 5x + 4| = 0$ (iii) $|x^2 - 6x + 8| = 4$

(iv) $|3x^2 - 7x + 2| = x^2 - x + 1$ (v) $|x^2 - 4| < 5$ (vi) $|x^2 - 3x + 2| > 4$

(vii) $|x^2 - 5x + 6| \le x + 2$ (viii) $|2x^2 - 3x - 5| < 4$

3.4 Solution of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic equation. We shall discuss the solutions of the rational, radical and exponential equations.

3.4.1 Rational Equations Reducible to the Quadratic Equation

A rational equation is an equation containing one or more rational expressions, where rational expressions typically contain a variable in the denominator.

Example 7: Solve $\frac{1}{x} + \frac{2}{x+1} = 1,\ x \ne 0,\ x \ne -1$

Solution: $\frac{1}{x} + \frac{2}{x+1} = 1$

Multiplying both sides by $x(x+1)$, we have

$$(x+1) + 2x = x(x+1)$$

$$x + 1 + 2x = x^2 + x$$

$$3x + 1 = x^2 + x$$

$$x^2 + x - 3x - 1 = 0$$

$$x^2 - 2x - 1 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{4+4}}{2}$$

$$x = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Hence, Solution Set $= \{1 \pm \sqrt{2}\}$

3.4.2 Radical Equations Reducible to the Quadratic Equation

Equations involving radical expressions of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical-free equation but the new equation has solutions that are not solutions of the original radical equation. Such extra solutions (roots) are called extraneous roots.

Example 8: Solve $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$

Solution: $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$

Squaring both sides, we get

$$x+8+x+3+2\sqrt{x+8}\sqrt{x+3} = 12x+13$$

$$2\sqrt{x+8}\sqrt{x+3} = 10x+2$$

$$\Rightarrow \quad \sqrt{(x+8)(x+3)} = 5x+1$$

Squaring again, we have

$$x^2+11x+24 = 25x^2+10x+1$$

$$\Rightarrow \quad 24x^2-x-23=0$$

$$\Rightarrow \quad (24x+23)(x-1)=0$$

$$x=-\frac{23}{24} \text{ or } x=1$$

On checking we find that $-\frac{23}{24}$ is an extraneous root. Hence solution set = $\{1\}$

3.5 Real World Problems of Quadratic Equations and Inequalities

We shall now proceed to solve the problems which, when expressed symbolically, lead to quadratic equations in one or two variables.

In order to solve such problems, we must:

i. Suppose the unknown quantities to be x or y etc.
ii. Translate the problem into symbols and form the equations satisfying the given conditions.

The method of solving the problems will be illustrated through the following examples:

Example 9: The length of a room is 3 metres greater than its breadth. If the area of the room is 180 square metres, find length and the breadth of the room.

Solution: Let the breadth of room = x metres

and the length of room = $(x+3)$ metres

$\therefore$ Area of the room = $x(x+3)$ square metres

By the given condition, we have

$$x(x+3) = 180 \qquad \ldots(i)$$

$$\Rightarrow \quad x^2 + 3x - 180 = 0 \qquad \ldots(ii)$$

$$\Rightarrow \quad (x+15)(x-12) = 0$$

$$\therefore \quad x = -15 \quad \text{or} \quad x = 12$$

As breadth cannot be negative so $x = -15$ is not admissible.

When $x = 12$, we get $x + 3 = 12 + 3 = 15$

Hence breadth of the room = 12 metres and length of the room = 15 metres.

Example 10: A company manufactures laptops and its weekly profit function (in thousands of dollars) is $P(x) = -x^2 + 2x + 3$, where x is the number of laptops produced (in hundreds). Find the range of production levels where the company makes at least \$4,000 profit.

Solution: Here $P(x) \geq 4$

$$-x^2 + 2x + 3 \geq 4$$

$$-x^2 + 2x + 3 - 4 \geq 0$$

$$-x^2 + 2x - 1 \geq 0$$

$$x^2 - 2x + 1 \leq 0$$

$$(x-1)^2 \leq 0$$

This only holds true when $(x-1)^2 = 0 \Rightarrow x = 1$

The company makes exactly \$4,000 profit when 100 laptops are produced (since $x = 1$ means 100 laptops). There is no production level where profit is more than \$4,000.

EXERCISE 3.2

1. Solve the following equations:

(i) $\dfrac{1}{3x} + \dfrac{4x}{6} = 1,\ x \neq 0$

(ii) $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{5}{2};\ x \neq -1, 0$

(iii) $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{7}{x+5};\ x \neq -1, -2, -5$

(iv) $\dfrac{a}{ax-1} + \dfrac{b}{bx-1} = a + b;\ x \neq \dfrac{1}{a}, \dfrac{1}{b}$

(v) $\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1} = 2,\ x \neq 1,\ x \neq -1$

(vi) $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$

(vii) $\sqrt{2x+8} + \sqrt{x+5} = 7$

(viii) $\sqrt{3x+4} = 2 + \sqrt{2x-4}$

(ix) $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

(x) $\sqrt{x+5} - \sqrt{x-3} = 2$

2 A farmer bought some sheep for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

3. A man sold his stock of eggs for Rs. 2400. If he had 2 dozen more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

4. A cyclist travelled 48 km at a uniform speed. If he had travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48 km?

5. To do a piece of work, Abdullah takes 10 days more than Abdul Hadi. Together they finish the work in 12 days. How long would Abdul Hadi take to finish it alone?

6. The braking distance (in metres) of a car is modeled by:

$d(s) = 0.02s^2 + 0.1s$, where s is the speed of car in km/h

If the maximum safe braking distance is 50 metres, find the range of speed where braking is safe.

7. A rocket follows the height function $h(t) = -5t^2 + 20t + 30$, where $h(t)$ is the height in metres and t is the time in seconds. Find the time interval during which the rocket is at least 40 metres above the ground.

Unit 4 Matrices & Determinants

INTRODUCTION

This unit introduces the fundamental concepts and operations of matrices, equipping students with the skills to perform matrix addition, subtraction and multiplication involving both real and complex entries. It explores the essential properties of determinants and provides techniques for evaluating the determinant of a 3×3 matrix using cofactors and determinant properties. Students will learn to apply row operations to determine the inverse and rank of matrices, as well as distinguish between consistent and inconsistent systems of linear equations through practical examples. The unit further explores into solving systems of linear equations, both homogeneous and non-homogeneous, using advanced methods such as matrix inversion, Cramer's Rule and Gaussian elimination. Emphasis is placed on the real-world applications of matrices in diverse fields such as graphic design, cryptography, data encryption, geometric transformations and highlighting the importance and versatility of matrix algebra in solving complex, practical problems.

4.1 Matrix

While solving linear systems of equations, a new notation was introduced to reduce the amount of writing. For this new notation the word *matrix* was first used by the English mathematician James Sylvester (1814 – 1897). Arthur Cayley (1821 – 1895) developed the theory of matrices and used them in the linear transformations. Now-a-days, matrices are used in high speed computers and also in other various disciplines.
The concept of determinants was used by Chinese and Japanese mathematicians but the Japanese mathematician Seki Kowa (1642–1708) and the German Mathematician Gottfried Wilhelm Leibniz (1646–1716) are credited for the invention of determinants. G. Cramer (1704–1752) employed the determinants successfully for solving the systems of linear equations.
A rectangular array of numbers enclosed by a pair of bracket is called a matrix such as:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{(i)} \qquad \text{or} \qquad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix} \quad \text{(ii)}$$

The horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. The numbers used in rows or columns are said to be the **entries** or **elements** of the matrix.

The matrix in (i) has two rows and three columns while the matrix in (ii) has four rows and three columns. Note that the number of the elements of the matrix in (ii) is $4\times 3=12$. Now the general definition of a matrix is:

Generally, a bracketed rectangular array of $m\cdot n$ elements a_{ij}(1, 2, 3, …, m; j = 1, 2, 3, …, n), arranged in m rows and n columns such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \qquad \text{(iii)}$$

is called an m by n matrix (written as $m\times n$ matrix), where $m\times n$ is called the **order** of the matrix in (iii). The matrices are usually represented by the capital letters such as A, B, C, X, Y, etc., and small letters such as $a, b, c, l, m, n,$ *or* $a_{11}, a_{12}, a_{13}, \ldots$, etc., are used to indicate the entries of the matrices.

Let the matrix in (iii) be denoted by A. The ith row and the jth column of A are indicated in the following tabular representation of A.

$$\begin{array}{c} \\ \\ \\ A= \\ i\text{th row} \rightarrow \\ \\ \\ \end{array} \overset{\displaystyle j\text{th column} \atop \displaystyle \downarrow \qquad\qquad}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}} \qquad \text{(iv)}$$

The elements of the ith row of A are $a_{i1}\ \ a_{i2}\ \ a_{i3} \ldots a_{ij} \ldots a_{in}$ while the elements of the jth column of A are $a_{1j}\ \ a_{2j}\ \ a_{3j} \ldots a_{ij} \ldots a_{mj}$. We note that a_{ij} is the element of the ith row and jth column of A. The double subscripts are useful to name the elements of the matrices. For example, the element 7 is at a_{23} position in the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$.

For convenience, we shall write the matrix A as:

$A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$, for $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$, where a_{ij} is the element of the ith row and jth column of A.

The elements (entries) of matrices need not always be numbers but in the study of matrices, we shall take the elements of the matrices from R or from C.

Note: The matrix A is called real matrix if all of its elements are real.

Row Matrix or Row vector: A matrix, which has only one row, i.e., $1 \times n$ matrix of the form $\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \ldots & a_{in} \end{bmatrix}$ is said to be a row matrix or a row vector.

Column Matrix or Column Vector: A matrix which has only one column i.e., an $m \times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a column matrix or a column vector.

For example $[1 \ -1 \ \ 3 \ \ 4]$ is a row matrix having 4 columns and $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix: If $m \neq n$, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular matrix. For example;

$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & 4 \\ 3 & -1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ are rectangular matrices of orders 2×3 and 4×3 respectively.

Square Matrix: If $m = n$, then the matrix of order $m \times n$ is said to be a square matrix of order n or m. i.e., the matrix which has the same number of rows and columns is called a square matrix. For example: $\begin{bmatrix} 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ are square matrices of orders 1, 2 and 3 respectively.

Let $A = [a_{ij}]$ be a square matrix of order n, then the entries $a_{11}, a_{22}, a_{33}, ..., a_{nn}$ form the **principal diagonal** for the matrix A and the entries $a_{1n}, a_{2\,n-1}, a_{3\,n-2}, ..., a_{n-1\,2}, a_{n1}$ form the secondary diagonal for the matrix A. For example, in the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, the entries of the principal diagonal are $a_{11}, a_{22}, a_{33}, a_{44}$ and the entries of the secondary diagonal are $a_{14}, a_{23}, a_{32}, a_{41}$.

The principal diagonal of a square matrix is also called the **leading diagonal** or **main diagonal** of the matrix.

Diagonal Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a diagonal matrix. The matrices

$$[7], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ are diagonal matrices.}$$

Scalar Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the matrix A is called a scalar matrix of order n. For example:

$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ $(a \neq 0)$ and $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of order 2, 3 and 4 respectively.

Unit Matrix or Identity Matrix: Let $A = [a_{ij}]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq$ j and $a_{ij} = 1$ for all $i =$ j, then the matrix A is called a *unit matrix* or *identity matrix* of order n. We denote such a matrix by I_n or simply I and it is of the form:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix of order 3 is denoted by I_3, that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix: A square or rectangular matrix whose each element is zero, is called a **null** or **zero matrix**. An $m \times n$ matrix with all its elements *equal* to zero, is denoted by $O_{m\times n}$. Null matrices may be of any order. Here are some examples:

$$[0], [0 \quad 0 \quad 0], \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note:
O may be used to denote null matrix of any order if there is no confusion.

are null matrices of order 1, 1 × 3, 2 × 3, 2 × 2, 3 × 1, 3 × 4 respectively.

Equal Matrices: Two matrices of the same order are said to be equal if they have same order and their corresponding entries are equal. For example, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal, i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1, 2, 3, ..., m$, $j = 1, 2, 3, ..., n$. In other words, A and B represent the same matrix.

Transpose of a Matrix: If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A. It is denoted by A^t. Let $A = [a_{ij}]_{m\times n}$,then the transpose of A is defined as:

$$A^t = [a'_{ij}]_{n\times m} \text{ where } a'_{ij} = a_{ji} \text{ for } i = 1, 2, 3, \ldots, n \text{ and } j = 1, 2, 3, \ldots, m$$

For example, if $B = [b_{ij}]_{3\times 4} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then

$$B^t = [b'_{ij}]_{4\times 3} \text{ where } b'_{ij} = b_{ji} \text{ for } i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3 \text{ i.e.,}$$

$$B^t = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \\ b'_{41} & b'_{42} & b'_{43} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$$

Note that the 2nd row of B has the same entries respectively as the 2nd column of B^t and the 3rd row of B^t has the same entries respectively as the 3rd column of B etc.

4.2 Matrix Operations

Matrix operations involve various techniques and procedures applied to matrices. These operations are foundational in linear algebra and have applications in numerous fields such as computer graphics, physics, statistics, etc. Here are some key matrix operations:

4.2.1 Addition of Matrices

Two matrices are conformable for addition if they are of the same order.

The sum $A + B$ of two $m \times n$, matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ is the $m \times n$ matrix $C = \left[c_{ij}\right]$ formed by adding the corresponding entries of A and B together. In symbols, we write as $C = A + B$, that is:

$$\left[c_{ij}\right] = \left[a_{ij} + b_{ij}\right] \text{ where } c_{ij} = a_{ij} + b_{ij} \text{ for } i = 1, 2, 3, ..., m;\ j = 1, 2, 3, ..., n$$

4.2.2 Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then we define subtraction of B from A as:

$$\begin{aligned} A - B &= A + (-B) \\ &= [a_{ij}] + [-b_{ij}] = [a_{ij} + (-b_{ij})] = [a_{ij} - b_{ij}] \text{ for } i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n \end{aligned}$$

Thus, the matrix $A - B$ is formed by subtracting each entry of B from the corresponding entry of A.

Example 1: If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$, then show that

$$(A + B)^t = A^t + B^t$$

Solution:

$$A + B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+(-1) & -1+3 & 2+1 \\ 3+1 & 1+3 & 2+(-1) & 5+4 \\ 0+3 & -2+1 & 1+2 & 6+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix}$$

and $$(A + B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \qquad \text{(i)}$$

Taking transpose of A and B, we have

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} \text{ and } B^t = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii), we have $(A+B)^t = A^t + B^t$

4.2.3 Scalar Multiplication

If $A = [a_{ij}]$ is $m \times n$ matrix and k is a real or complex number, then the product of k and A, denoted by kA, is the matrix formed by multiplying each entry of A by k, that is

$kA = [ka_{ij}]$

Obviously, order of kA is $m \times n$.

> **Note:**
> If n is a positive integer, then $A + A + A + \ldots$ to n terms $= nA$.

4.2.4 Multiplication of two Matrices

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be a 2×3 matrix and $B = [b_{ij}]$ be a 3×2 matrix, then the product AB is defined to be the 2×2 matrix C whose element c_{ij} is the sum of products of the corresponding elements of the ith row of A with elements of jth column of B. For example, the element c_{21} of C is shown in the figure (A), that is

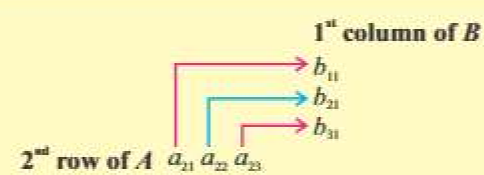


Figure (A)

$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$. Thus

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix} \quad \text{(i)}$$

Similarly $BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix} \quad \text{(ii)}$$

From (i) and (ii), AB and BA are calculated their orders are 2×2 and 3×3 respectively.

Note 1. In general, $AB \neq BA$

Note 2. If the product AB is defined, then the order of the product can be illustrated as given below:

Order of A	$m\times n$
Order of B	$n\times p$
Order of AB	$m\times p$

Example 2: If $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & 3 \\ -1 & -4 & 6 \\ 0 & -5 & 5 \end{bmatrix}$, then compute A^2B.

Solution: $A^2 = A\cdot A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 4-1+0 & -2-2+0 & 0+3+0 \\ 2+2-3 & -1+4-6 & 0-6+6 \\ 2+2-2 & -1+4-4 & 0-6+4 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -3 & 0 \\ 2 & -1 & -2 \end{bmatrix}$$

$$\therefore \quad A^2B = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -3 & 0 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ -1 & -4 & 6 \\ 0 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4+0 & -6+16-15 & 9-24+15 \\ 2+3+0 & -2+12+0 & 3-18+0 \\ 4+1+0 & -4+4+10 & 6-6-10 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 0 \\ 5 & 10 & -15 \\ 5 & 10 & -10 \end{bmatrix}$$

Note: Powers of square matrices are defined as:

$A^2 = A\times A, A^3 = A\times A\times A$

$A^n = A\times A\times A\times \cdots$ to n factors.

4.3 Properties of Matrix Addition, Scalar Multiplication and Matrix Multiplication

If A, B and C are conformable for the indicated sum or product of matrices and c and d are scalars, then following properties are true:

(i) Commutative property w.r.t. addition: $A+B=B+A$

(ii) Associative property w.r.t. addition: $(A+B)+C=A+(B+C)$

(iii) Associative property of scalar multiplication: $(cd)A=c(dA)$

(iv) Existence of additive identity: $A+O = O+A=A$ $\begin{pmatrix} O \text{ is null matrix and} \\ A \text{ is a square matrix} \end{pmatrix}$

(v) Existence of multiplicative identity: $IA=AI=A$ (I is unit/identity matrix)

(vi) Distributive property w.r.t scalar multiplication:

(a) $c(A+B)=cA+cB$ (b) $(c+d)A=cA+dA$

(vii) Associative property w.r.t. multiplication: $A(BC)=(AB)C$

(viii) Left distributive property: $A(B+C)=AB+AC$

(ix) Right distributive property: $(A+B)C=AC+BC$

(x) $c(AB)=(cA)B=A(cB)$

Example 3: Find AB and BA if $A=\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}$

Solution: $AB=\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 2\times1+0\times2+1\times1 & 2\times(-1)+0\times3+1\times(-2) & 2\times0+0\times(-1)+1\times3 \\ 1\times1+4\times2+2\times1 & 1\times(-1)+4\times3+2\times(-2) & 1\times0+4\times(-1)+2\times3 \\ 3\times1+0\times2+6\times1 & 3\times(-1)+0\times3+6\times(-2) & 3\times0+0\times(-1)+6\times3 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & -4 & 3 \\ 11 & 7 & 2 \\ 9 & -15 & 18 \end{bmatrix} \qquad \text{(i)}$$

$$BA=\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2+(-1)\times 1+0\times 3 & 1\times 0+(-1)\times 4+0\times 0 & 1\times 1+(-1)\times 2+0\times 6 \\ 2\times 2+3\times 1+(-1)\times 3 & 2\times 0+3\times 4+(-1)\times 0 & 2\times 1+3\times 2+(-1)\times 6 \\ 1\times 2+(-2)\times 1+3\times 3 & 1\times 0+(-2)\times 4+3\times 0 & 1\times 1+(-2)\times 2+3\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -1 \\ 4 & 12 & 2 \\ 9 & -8 & 15 \end{bmatrix} \qquad \text{(ii)}$$

Note:
Matrix multiplication is not commutative in general.

Thus, from (i) and (ii), $AB \neq BA$

EXERCISE 4.1

1. If $A=[a_{ij}]_{3\times 4}$, then show that

   (i) $I_3A=A$ (ii) $AI_4=A$

2. If $A=\begin{bmatrix} 0 & -1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} 1 & 0 & -2 \\ -1 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, then find

   (i) $A-B$ (ii) B – C (iii) $(A-B)-C$ (iv) $A-(B-C)$

3. If A and B are square matrices of the same order, then explain why in general:

   (i) $(A+B)^2 \neq A^2+2AB+B^2$ (ii) $(A-B)^2 \neq A^2-2AB+B^2$

   (iii) $(A+B)(A-B) \neq A^2-B^2$

4. If $A=\begin{bmatrix} -1 & 2 & 3 & 0 \\ 1 & 0 & 2 & -2 \\ -3 & 5 & 3 & -1 \end{bmatrix}$, then find AA', $A'A$ and $\left(A'\right)'$.

5. Solve the following matrix equations for X:

   (i) $2X-3A=B$ if $A=\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

   (ii) $A^2-5A+4I-X=0$ if $A=\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

4.4 Determinants

The determinants of square matrices of order $n \geq 3$, can be written by following the pattern. For example, if $n=3$

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the determinant of $A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Now our aim is to compute the determinants of matrices of various orders.

4.4.1 Minor and Cofactor of an Element of a Matrix or its Determinant

Minor of an Element: Let us consider a square matrix A of order n, then the minor of an element a_{ij}, denoted by M_{ij} is the determinant formed by deleting the ith row and the jth column of $A(\text{or}|A|)$.

For example, consider a square matrix A of order 3, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

To find the minor of the element a_{12}, delete the first row and second column of A

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, that is $M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of an Element: The cofactor of an element a_{ij} of a square matrix A denoted by A_{ij} is defined by $A_{ij}=(-1)^{i+j}M_{ij}$

For example, $A_{12}=(-1)^{1+2}M_{12}=(-1)^3\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

4.4.2 Determinant of a Square Matrix of Order $n = 3$

If A is a matrix of order 3, that is, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$ for $i = 1, 2, 3$

or $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$ for $j = 1, 2, 3$

For example, for $i=1, j=1$ and $j=2$, we have

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (i)

or $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (ii)

or $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ (iii)

(iii) can be written as: $|A| = a_{12}(-1)^{1+2}M_{12} + a_{22}(-1)^{2+2}M_{22} + a_{32}(-1)^{3+2}M_{32}$

i.e., $|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$ (iv)

Similarly (i) can be written as $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ (v)

Putting the values of M_{11}, M_{12} and M_{13} in (v), we obtain

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$ (vi)

or $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$ (vii)

Equation (vii) is the required expansion of determinant of square matrix of order 3.

Example 4: Evaluate the determinant if $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Solution: $|A| = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{vmatrix}$

using $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, we get

$$|A| = 1\begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

$$= 1[6-1(-3)] + 2[(-2)(2)-(1)(4)] + 3[(-2)(-3)-12]$$

$$= (6+3) + 2(-4-4) + 3(6-12) = 9 - 16 - 18 = -25$$

Example 5: Find the cofactors A_{12}, A_{22} and A_{32} of $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ and find $|A|$.

Solution: We first find M_{12}, M_{22} and M_{32},

$$M_{12} = \begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} = -4-4 = -8 \;;\; M_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2-12 = -10$$

and $M_{32} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1-(-6) = 7$

Thus $A_{12} = (-1)^{1+2}M_{12} = (-1)(-8) = 8;\quad A_{22} = (-1)^{2+2}M_{22} = 1(-10) = -10$

$A_{32} = (-1)^{3+2}M_{32} = (-1)(7) = -7$

and $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = (-2)8 + 3(-10) + (-3)(-7)$

$= -16 - 30 + 21 = -25$

4.4.3 Properties of Determinants

i. For a square matrix A, $|A| = |A^t|$

ii. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.

iii. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.

iv. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.

v. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in R$, then the determinant of the resulting matrix is $k|A|$.

vi. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$|B| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

vii. If any row (column) of a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change.

viii. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Note: We shall define triangular matrices on following pages

Example 6: Without expansion, show that $\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = 0$

Solution: Adding the entries of C_3 to the corresponding entries of C_2.

$$= \begin{vmatrix} x & a+b+c+x & b+c \\ x & a+b+c+x & c+a \\ x & a+b+c+x & a+b \end{vmatrix}$$

$$= x(a+b+c+x)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix} \quad \begin{pmatrix}\text{by taking } x \text{ from } C_1 \text{ and } (a+b+c+x) \\ \text{common from } C_2\end{pmatrix}$$

$$= x(a+b+c+x)\cdot 0 \quad (\because C_1 \text{ and } C_2 \text{ are identical})$$

$$= 0$$

4.5 Adjoint and Inverse of a Square Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix of co-factors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$,

and adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Inverse of a Square Matrix of Order $n \geq 3$: Let A be a non singular ($|A| \neq 0$) square matrix of order n. If there exists a matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.

Thus, $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

If A *is* non singular matrix then

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

Example 7: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

Solution: We first find the cofactor of the elements of A.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 1.(2+1) = 3, \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1) = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 1.(0-2) = -2, \quad A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = (-1)(0+2) = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1.(1-2) = -1, \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = (-1)(-1-0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 1.(0-4) = -4, \qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1-0) = -1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1.(2-0) = 2$$

Thus $$[A_{ij}]_{3\times 3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$$

and $$\text{adj } A = [A'_{ij}]_{3\times 3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} \qquad (\because A'_{ij} = A_{ji} \text{ for } i, j = 1, 2, 3)$$

Since $$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(3) + 0(1) + 2(-2)$$
$$= 3 + 0 - 4 = -1$$

So $$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{-1}\begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

EXERCISE 4.2

1. Evaluate the following determinants:

$$\text{(i)} \begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix} \qquad \text{(ii)} \begin{vmatrix} 5 & 2 & -3 \\ 3 & 10 & -1 \\ -2 & 1 & -2 \end{vmatrix} \qquad \text{(iii)} \begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 5 \\ -2 & 5 & 6 \end{vmatrix}$$

$$\text{(iv)} \begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix} \qquad \text{(v)} \begin{vmatrix} 1 & 20 & -2 \\ -1 & -1 & -3 \\ 2 & 4 & -1 \end{vmatrix} \qquad \text{(vi)} \begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$$

2. Without expansion show that:

$$\text{(i)} \begin{vmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{vmatrix} = 0 \qquad \text{(ii)} \begin{vmatrix} 5 & 6 & -1 \\ 2 & 2 & 0 \\ 2 & -8 & 10 \end{vmatrix} = 0 \qquad \text{(iii)} \begin{vmatrix} -a & 0 & b \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix} = 0$$

(iv) $\begin{vmatrix} l & m+n & 1 \\ m & n+l & 1 \\ n & l+m & 1 \end{vmatrix} = 0$ (v) $\begin{vmatrix} 2 & 1 & 3x \\ 2 & 3 & 9x \\ 3 & 5 & 15x \end{vmatrix} = 0$ (vi) $\begin{vmatrix} 1 & p^2 & \frac{p}{qr} \\ 1 & q^2 & \frac{q}{rp} \\ 1 & r^2 & \frac{r}{pq} \end{vmatrix} = 0$

(vii) $\begin{vmatrix} p-q & q-r & r-p \\ q-r & r-p & p-q \\ r-p & p-q & q-r \end{vmatrix} = 0$ (viii) $\begin{vmatrix} yz & zx & xy \\ x & y & z \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix} = 0$

(ix) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ (x) $\begin{vmatrix} 2a & a+b & a+c \\ 2b & 2b & b+c \\ 2c & b+c & 2c \end{vmatrix} = 0$

3. Show that:

(i) $\begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix} = 25\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$

(iii) $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ (iv) $\begin{vmatrix} m+n & l & l \\ m & n+l & m \\ n & n & l+m \end{vmatrix} = 4lmn$

(v) $\begin{vmatrix} y & -1 & x \\ x & y & 0 \\ 1 & x & y \end{vmatrix} = x^3 + y^3$ (vi) $\begin{vmatrix} r\cos\theta & 1 & -\sin\theta \\ 0 & 1 & 0 \\ r\sin\theta & 0 & \cos\theta \end{vmatrix} = r$

(vii) $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

(viii) $\begin{vmatrix} a+\lambda & a & a \\ b & b+\lambda & b \\ c & c & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

(ix) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(x) $\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x+y+z)(x-y)(y-z)(z-x)$

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 0 \\ -2 & -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -2 & 5 \\ -3 & -1 & 4 \\ -2 & -1 & 2 \end{bmatrix}$, then find:

(i) A_{13}, A_{23}, A_{33} and $|A|$ (ii) B_{31}, B_{32}, B_{33} and $|B|$

5. Find values of x if:

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & -3 & -4 \\ x & 1 & 0 \end{vmatrix} = -30$ (ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -3 & x \end{vmatrix} = 0$ (iii) $\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

6. Show that: $\begin{vmatrix} x & 2 & 2 \\ 2 & x & 2 \\ 2 & 2 & x \end{vmatrix} = (x+4)(x-2)^2$.

7. Find $|AA'|$ and $|A'A|$ if: (i) $A = \begin{bmatrix} -3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

8. If A is a square matrix of order 3, then show that $|kA| = k^3|A|$.

9. Find the values of λ if A and B are singular.

$A = \begin{bmatrix} 4 & 2 & 3 \\ 7 & \lambda & 6 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 & 5 \\ 1 & -2 & 1 \\ 2 & \lambda & 0 \end{bmatrix}$

10. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{bmatrix}$ and show that $A^{-1}A = I_3$

11. Verify that $(AB)^t = B^t A^t$ if:

(i) $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -3 & -2 \\ 0 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

4.6 Elementary Row Operations on a Matrix

Usually, a given system of linear equations is reduced to a simple equivalent system by applying elementary operations which are stated as below:

(i) Interchanging two equations.
(ii) Multiplying an equation by a non-zero number.
(iii) Adding a multiple of one equation to another equation.

Corresponding to these three elementary operations, the following elementary row operations are applied to matrices to obtain equivalent matrices.

(i) Interchanging two rows
(ii) Multiplying a row by a non-zero number
(iii) Adding a multiple of one row to another row.

Note: Matrices A and B are equivalent if B can be obtained by applying in turn a finite number of row operations on A.

Notations that are used to represent row operations for I to III are given below:

Interchanging R_i and R_j is expressed as $R_i \leftrightarrow R_j$.

k times R_i is denoted by $kR_i \rightarrow R_i'$

Adding k times R_j to R_i is expressed as $R_i + kR_j \rightarrow R_i'$

(R_i' is the new row obtained after applying the row operation).

For equivalent matrices A and B, we write $A \underset{\sim}{R} B$.

If $A \underset{\sim}{R} B$ then $B \underset{\sim}{R} A$

Upper Triangular Matrix: A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if all elements below the principal diagonal are zero, that is,

$$a_{ij} = 0 \text{ for all } i > j$$

Lower Triangular Matrix: A square matrix $A = [a_{ij}]$ is said to be lower triangular matrix if all elements above the principal diagonal are zero, that is,

$$a_{ij} = 0 \text{ for all } i < j$$

Triangular Matrix: A square matrix A is named as triangular matrix whether it is upper triangular or lower triangular. For example, the matrices

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 1 & 5 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix}$ are triangular matrices of order 3 and 4 respectively.

The first matrix is upper triangular while the second is lower triangular.

Remember!

Diagonal matrices are both upper triangular and lower triangular.

4.7 Echelon and Reduced Echelon Forms of Matrices

In any non-zero row of a matrix, the first non-zero entry is called the leading entry of that row.

Echelon Form of a Matrix

An $m \times n$ matrix A is called in echelon form if:

(i) The number of zeros before the leading entry is greater than the number zeros in the preceding row.

(ii) Every non-zero row in A precedes every zero row (if any).

(iii) The first non-zero entry (or leading entry) in each row is 1.

The matrices $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form

Reduced Echelon Form of a Matrix: An $m \times n$ matrix A is said to be in reduced (row) echelon form if the first non-zero entry (or leading entry) in R_i lies in C_j, then all other entries of C_j are zero.

The matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Example 8: Reduce $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ to (row) echelon and reduced (row) echelon form.

Solution: $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$, $R_1 \sim R_2$ $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & -1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & -3\\0 & 5 & -5 & 15\\0 & 4 & -3 & 11\end{bmatrix}\ \text{By } R_2+(-2)R_1\to R_2' \text{ and } R_3+(-3)R_1\to R_3' \quad \underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & -3\\0 & 1 & -1 & 3\\0 & 4 & -3 & 11\end{bmatrix}\ \frac{1}{5}R_2\to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & -3\\0 & 1 & -1 & 3\\0 & 0 & 1 & -1\end{bmatrix}\ R_3+(-4)R_2\to R_3' \quad \underset{\sim}{R}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & -1 & 3\\0 & 0 & 1 & -1\end{bmatrix}\ R_1+1.R_2\to R_1'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -1\end{bmatrix}\ \text{By } R_1+(-1)R_3\to R_1' \text{ and } R_2+1.R_3\to R_2'$$

Thus $\begin{bmatrix}1 & -1 & 2 & -3\\0 & 1 & -1 & 3\\0 & 0 & 1 & -1\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -1\end{bmatrix}$ are (row) echelon and reduced (row) echelon forms of the given matrix respectively.

Inverse of a Matrix: Let A be a non-singular matrix. If the application of elementary row operations on $A\vdots I$ in succession reduces A to I, then the resulting matrix is $I\vdots A^{-1}$.

Example 9: Find the inverse of the matrix $A=\begin{bmatrix}2 & 5 & -1\\3 & 4 & 2\\1 & 2 & -2\end{bmatrix}$

Solution: $|A|=\begin{vmatrix}2 & 5 & -1\\3 & 4 & 2\\1 & 2 & -2\end{vmatrix}=2(-8-4)-5(-6-2)-1(6-4)=-24+40-2=40-26=14$

As $|A|\neq 0$, so A is non-singular.

Appending I_3 on the right of the matrix A, we have $\begin{bmatrix}2 & 5 & -1 & \vdots & 1 & 0 & 0\\3 & 4 & 2 & \vdots & 0 & 1 & 0\\1 & 2 & -2 & \vdots & 0 & 0 & 1\end{bmatrix}$

Interchanging R_1 and R_3 we get,

$$\begin{bmatrix}1 & 2 & -2 & \vdots & 0 & 0 & 1\\3 & 4 & 2 & \vdots & 0 & 1 & 0\\2 & 5 & -1 & \vdots & 1 & 0 & 0\end{bmatrix}\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2 & \vdots & 0 & 0 & 1\\0 & -2 & 8 & \vdots & 0 & 1 & -3\\0 & 1 & 3 & \vdots & 1 & 0 & -2\end{bmatrix}\ \text{By } R_2+(-3)R_1\to R_2' \text{ and } R_3+(-2)R_1\to R_3'$$

By $-\frac{1}{2}R_2 \rightarrow R_2'$, we get

$$\begin{bmatrix} 1 & 2 & -2 & \vdots & 0 & 0 & 1 \\ 0 & 1 & -4 & \vdots & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & \vdots & 1 & 0 & -2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 6 & \vdots & 0 & 1 & -2 \\ 0 & 1 & -4 & \vdots & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 7 & \vdots & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \begin{matrix} \text{By } R_3 + (-1)R_2 \rightarrow R_3' \\ \text{and } R_1 + (-2)R_2 \rightarrow R_1' \end{matrix}$$

By $\frac{1}{7}R_3 \rightarrow R_3'$, we have

$$\begin{bmatrix} 1 & 0 & 6 & \vdots & 0 & 1 & -2 \\ 0 & 1 & -4 & \vdots & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & \vdots & -\frac{6}{7} & \frac{4}{7} & 1 \\ 0 & 1 & 0 & \vdots & \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \begin{matrix} \text{By } R_1 + (-6)R_3 \rightarrow R_1' \\ \text{and } R_2 + 4R_3 \rightarrow R_2' \end{matrix}$$

Thus, the inverse of A is $\begin{bmatrix} -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix}$

Rank of a Matrix: Let A be a non-zero matrix. If r is the number of non-zero rows when it is reduced to the echelon form, then r is called the rank of the matrix A.

Example 10: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix}$ By $R_2 + (-2)R_1 \rightarrow R_2'$ and $R_3 + (-3)R_1 \rightarrow R_3'$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 4 & 6 & -2 \end{bmatrix} \text{By } \frac{1}{2}R_2 \rightarrow R_2' \quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{By } R_3 + (-4)R_2 \rightarrow R_3'$$

As the number of non-zero rows is 2 when the given matrix is reduced to echelon form, therefore, the rank of the given matrix is 2.

4.8 System of Non-Homogeneous Linear Equations

Three linear equations in three variables such as:

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} \qquad \text{(i)}$$

is called a system of non-homogeneous linear equations in the three variables x, y and z, if constant terms d_1, d_2 and d_3 are not all zero.

Consistent: A system of linear equations is said to be consistent if the system has a unique solution or it has infinitely many solutions.

Inconsistent: A system of linear equations is said to be inconsistent if the system has no solution.

Now we will solve the system of non-homogeneous linear equations with the help of the following methods:

(i) Using reduced echelon form (ii) Using matrix inversion method

(iii) Using Cramer's rule

4.8.1 Reduced Echelon Form

There are following steps to solve a system of non-homogeneous linear equations (i):

(i) Convert to augmented matrix

i.e. $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

(ii) Convert to reduced echelon form (iii) Solve by back substitution

Example 11: Solve the following and explain a consistent and inconsistent system:

(i) $2x+5y-z=5$, $3x+4y+2z=11$, $x+2y-2z=-3$

(ii) $x+y+2z=1$, $2x-y+7z=11$, $3x+5y+4z=-3$

(iii) $x-y+2z=1$, $2x-6y+5z=7$, $3x+5y+4z=-3$

Solution: (i) The augmented matrix of the given system is $\left[\begin{array}{ccc|c} 2 & 5 & -1 & 5 \\ 3 & 4 & 2 & 11 \\ 1 & 2 & -2 & -3 \end{array}\right]$

We apply the elementary row operations to the above matrix to reduce it to the equivalent reduced (row) echelon form, that is,

$$\left[\begin{array}{ccc|c} 2 & 5 & -1 & 5 \\ 3 & 4 & 2 & 11 \\ 1 & 2 & -2 & -3 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -3 \\ 3 & 4 & 2 & 11 \\ 2 & 5 & -1 & 5 \end{array}\right] \quad \text{By } R_1 \leftrightarrow R_3$$

$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2 & \vdots & -3\\ 0 & -2 & 8 & \vdots & 20\\ 2 & 5 & -1 & \vdots & 5\end{bmatrix}$ By $R_2+(-3)R_1 \rightarrow R_2'$ $\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2 & \vdots & -3\\ 0 & -2 & 8 & \vdots & 20\\ 0 & 1 & 3 & \vdots & 11\end{bmatrix}$ By $R_3+(-2)R_1 \rightarrow R_3'$

By $-\frac{1}{2}R_2 \rightarrow R_2'$, we get

$$\begin{bmatrix}1 & 2 & -2 & \vdots & -3\\ 0 & 1 & -4 & \vdots & -10\\ 0 & 1 & 3 & \vdots & 11\end{bmatrix} \underset{\sim}{R} \begin{bmatrix}1 & 0 & 6 & \vdots & 17\\ 0 & 1 & -4 & \vdots & -10\\ 0 & 0 & 7 & \vdots & 21\end{bmatrix}$$ By $R_1+(-2)R_2 \rightarrow R_1'$ and $R_3+(-1)R_2 \rightarrow R_3'$

$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 6 & \vdots & 17\\ 0 & 1 & -4 & \vdots & -10\\ 0 & 0 & 1 & \vdots & 3\end{bmatrix}$ By $\frac{1}{7}R_3 \rightarrow R_3'$ $\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & \vdots & -1\\ 0 & 1 & 0 & \vdots & 2\\ 0 & 0 & 1 & \vdots & 3\end{bmatrix}$ *By* $R_1+(-6)R_3 \rightarrow R'_1$ *and* $R_2+4R_3 \rightarrow R'_2$

Thus, the solution is $x=-1, y=2$ and $z=3$, therefore the given system of linear equations has unique solution and it is consistent.

(ii) The augmented matrix of the given system is $\begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 2 & -1 & 7 & \vdots & 11\\ 3 & 5 & 4 & \vdots & -3\end{bmatrix}$

$\begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 2 & -1 & 7 & \vdots & 11\\ 3 & 5 & 4 & \vdots & -3\end{bmatrix} \underset{\sim}{R} \begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 0 & -3 & 3 & \vdots & 9\\ 0 & 2 & -2 & \vdots & -6\end{bmatrix}$ Adding $(-2)R_1$ to R_2 and $(-3)R_1$ to R_3.

We get, $\underset{\sim}{R}\begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 2 & -2 & \vdots & -6\end{bmatrix}$ By $-\frac{1}{3}R_2 \rightarrow R_2'$ $\underset{\sim}{R}\begin{bmatrix}1 & 0 & 3 & \vdots & 4\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix}$ By $R_1+(-1)R_2 \rightarrow R_1'$ and $R_3+(-2)R_2 \rightarrow R_3'$

The given system is reduced to equivalent system

$$x+3z=4$$
$$y-z=-3$$
$$0z=0$$

The equation $0z=0$ is satisfied by any value of z.

From the first and second equations, we get

$$x=-3z+4 \quad \text{(a)}$$

and $$y=z-3 \quad \text{(b)}$$

As z is arbitrary, so we can find infinitely many values of x and y from equations (a) and (b) or the given system, is satisfied by $x=4-3t, y=t-3$ and $z=t$ for any real value of t.

Thus, the given system has infinitely many solutions and it is consistent.

(iii) The augmented matrix of the system is $\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & -4 & 1 & \vdots & 5 \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix} \text{ Adding } (-2)R_1 \text{ to } R_2 \text{ and } (-3R_1) \text{ to } R_3.$$

We have,

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix} \text{ By } -\frac{1}{4}R_2 \to R_2' \quad \underset{\sim}{R} \begin{bmatrix} 1 & 0 & \frac{7}{4} & \vdots & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 0 & 0 & \vdots & 4 \end{bmatrix} \begin{matrix} \text{By } R_1 + 1.R_2 \to R_1' \\ \text{and } R_3 + (-8)R_2 \to R_3' \end{matrix}$$

Thus, the given system is reduced to the equivalent system

$$x + \frac{7}{4}z = -\frac{1}{4}$$
$$y - \frac{1}{4}z = -\frac{5}{4}$$
$$0z = 4$$

The third equation $0z = 4$ has no solution, so the system as a whole has no solution. Thus, the system is inconsistent.

Note: We see that in the case of the system (i), the (row) rank of the augmented matrix and the coefficient matrix of the system is the same, that is, 3 which is equal to the number of the variables in the system (i).

Thus, we observe that a linear system is consistent and has a unique solution if the rank of the coefficient matrix is the same as that of the augmented matrix of the system and equal to number of variables.

In the case of the system (ii), the rank of the coefficient matrix is the same as that of the augmented matrix of the system but it is 2 which is less than the number of variables in the system (ii).

Thus, we observe that a system is consistent and has infinitely many solutions if the ranks of the coefficient matrix and the augmented matrix of the system are equal but the rank is less than the number of variables in the system.

In the case of the system (iii), we see that the rank of the coefficient matrix is not equal to the rank of the augmented matrix of the system.

Thus, we observe that a system is inconsistent if the ranks of the coefficient matrix and the augmented matrix of the system are different.

4.8.2 Matrix Inversion Method

The matrix inversion method is a way to solve a system of linear equations using the inverse of a matrix.

Example 12: Use matrix inversion method to solve the system
$$\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ 2x_1 - 3x_2 + 2x_3 &= -6 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

Solution: The matrix form of the given system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$$

or $\quad AX = B \qquad \ldots$(i)

Where $\quad A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$

As $\quad |A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{vmatrix}$ By $R_2 + (-2)R_1 \rightarrow R_2'$

Expanding by R_2 we have

$$= (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1, \text{ that is,}$$

$|A| \neq 0$, so the inverse of A exists and (i) can be written as

$X = A^{-1}B \qquad \ldots$(ii)

Now we find adj A.

$$\Rightarrow \quad [A_{ij}]_{3\times 3} = \begin{bmatrix} -7 & 2 & 10 \\ 4 & -1 & -6 \\ -1 & 0 & 1 \end{bmatrix},$$

Cofactors are $\quad A_{11} = -7, A_{12} = 2, A_{13} = 10, A_{21} = 4$
$A_{22} = -1, A_{23} = -6, A_{31} = -1, A_{32} = 0, A_{33} = 1$

So $\quad \text{adj } A = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$

and $\quad A^{-1} = \dfrac{1}{|A|}\text{ adj}A = \dfrac{1}{-1}\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Thus, the solution set is $\{(x_1, x_2, x_3)\} = \{(1, 2, -1)\}$

4.8.3 Cramer's Rule

Consider the system of equations,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}\right\} \qquad \ldots(\text{ii})$$

These are three linear equations in three variables x_1, x_2, x_3 with coefficients and constant terms in the real field R. We write the above system of equations in matrix form as:

$$AX = B \qquad \ldots(\text{ii})$$

where $A = [a_{ij}]_{3\times 3}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

We know that the matrix equation (2) can be written as: $X = A^{-1}B$ (if A^{-1} exists)

We have already proved that $A^{-1} = \frac{1}{|A|}\text{adj } A$

and $\text{adj } A = [A'_{ij}]_{3\times 3} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ $(\because A'_{ij} = A_{ji})$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11}b_1 + A_{21}b_2 + A_{31}b_3 \\ A_{12}b_1 + A_{22}b_2 + A_{32}b_3 \\ A_{13}b_1 + A_{23}b_2 + A_{33}b_3 \end{bmatrix}$

i.e., $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{11}b_1 + A_{21}b_2 + A_{31}b_3}{|A|} \\ \dfrac{A_{12}b_1 + A_{22}b_2 + A_{32}b_3}{|A|} \\ \dfrac{A_{13}b_1 + A_{23}b_2 + A_{33}b_3}{|A|} \end{bmatrix}$$

Hence $$x_1 = \frac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{|A|} = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} \qquad \text{(iii)}$$

$$x_2 = \frac{b_1A_{12} + b_2A_{22} + b_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} \qquad \text{(iv)}$$

$$x_3 = \frac{b_1A_{13} + b_2A_{23} + b_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|} \qquad \text{(v)}$$

The method of solving the system with the help of results (iii), (iv) and (v) is often referred to as Cramer's Rule.

Example 13: Use Cramer's rule to solve the system. $$\left.\begin{aligned} 3x_1 + x_2 - x_3 &= -4 \\ x_1 + x_2 - 2x_3 &= -4 \\ -x_1 + 2x_2 - x_3 &= 1 \end{aligned}\right\}$$

Solution: Here $$|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+4) - 1.(-1-2) - 1.(2+1)$$

$$= 9 + 3 - 3 = 9$$

So, $$x_1 = \frac{\begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}}{9} = \frac{-4(-1+4) - 1(4+2) - 1(-8-1)}{9}$$

$$= \frac{-12-6+9}{9} = \frac{-9}{9} = -1$$

$$x_2 = \frac{\begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix}}{9} = \frac{3(4+2)+4(-1-2)-1(1-4)}{9}$$

$$= \frac{18-12+3}{9} = \frac{9}{9} = 1$$

$$x_3 = \frac{\begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix}}{9} = \frac{3(1+8)-1(1-4)-4(2+1)}{9} = \frac{27+3-12}{9} = \frac{18}{9} = 2$$

Hence $x_1 = -1,\ x_2 = 1,\ x_3 = 2$

Thus, the solution set is $\{(x_1, x_2, x_3)\} = \{(-1, 1, 2)\}$

4.9 System of Homogeneous Linear Equations

The system of following homogeneous linear equations:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}\right\} \qquad \ldots(\text{i})$$

is always satisfied by $x_1 = 0, x_2 = 0$ and $x_3 = 0$, so such a system is always consistent.

Trivial Solution: The solution (0, 0, 0) of the above homogeneous system is called the trivial solution.

Non-Trivial Solution: Any other solution of system (i) other than the trivial solution is called a non-trivial solution.

4.9.1 Solution of System of Homogeneous Linear Equations by Gaussian Elimination Method

Gaussian Elimination is a systematic method for solving systems of linear equations, named after the German mathematician Carl Friedrich Gauss. It involves performing a series of row operations on the system's augmented matrix to transform it into row-echelon form. Once the matrix is in this simplified form, the solution to the system can be determined through back substitution. This method is widely used due to its efficiency and clarity in solving linear systems.

Example 14: Solve the following system of equations by Gaussian Elimination method:

$$x + 2y + z = 0$$
$$2x + 3y + 4z = 0$$
$$4x + 3y + 2z = 0$$

Solution: The augmented matrix is

$$A_b = \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 2 & 3 & 4 & | & 0 \\ 4 & 3 & 2 & | & 0 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & -1 & 2 & | & 0 \\ 0 & -5 & -2 & | & 0 \end{bmatrix} By\, R_2 + (-2)R_1 \to R_2' \text{ and } R_3 + (-4)R_1 \to= R_3'$$

$$\Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & -5 & -2 & | & 0 \end{bmatrix} By\, (-1)R_2 \to R_2'$$

$$\Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & -12 & | & 0 \end{bmatrix} By\, R_3 + 5R_2 \to R_3'$$

$$\Rightarrow \underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} By \left(\frac{-1}{12}\right) R_3 \to R_3' \quad \text{(Rank of } A = 3 = \text{number of variables)}$$

The matrix is in row-echelon form.

By back-substitution, from the third row, $z = 0$.

from the second row: $y - 2z = 0$

$$y - 2(0) = 0$$
$$y = 0$$

From the first row, $x + 2y + z = 0$, substituting $y = 0$ and $x = 0$, we have

$$x + 2(0) + 0 = 0$$
$$x = 0$$

Thus, the system has only trivial solution, i.e., $(x, y, z) = (0, 0, 0)$.

Example 15: Solve the following system of equations using Gaussian Elimination Method.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 - x_3 &= 0 \end{aligned}$$

Solution: The argumented matrix is

$$A_b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 3 & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \text{ By } R_2 + (-1)R_1 \to R_2' \text{ and } R_3 + (-1)R_1 \to R_3'$$

$$\Rightarrow \quad \underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \text{ By} \left(-\frac{1}{2}\right)R_2 \to R_2'$$

$$\Rightarrow \quad \underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ By } R_3 + (-2)R_2 \to R_3'$$

(Rank of A < number of variables)

The matrix is in row-echelon form

Thus, the above system is reduced to the equivalent system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \quad \text{(i)} \\ x_2 - x_3 &= 0 \quad \text{(ii)} \\ 0x_3 &= 0 \end{aligned}$$

From (i) and (ii), we get

$$\begin{aligned} x_1 &= -x_2 - x_3 \quad \text{(iii)} \\ x_2 &= x_3 \end{aligned}$$

Substituting $x_2 = x_3$ in (iii), we get

$$x_1 = -x_3 - x_3 = 2x_3$$

$$\Rightarrow \quad x_1 = -2x_3 \quad \text{(iv)}$$

As x_3 is arbitrary, so we can find infinitely many values of x_1 and x_2 from (iii) and (iv) or the system is satisfied by $x_1 = -2t$, $x_2 = t$ and $x_3 = t$ for any value of t.

From above examples we observe that:

Rule – I: Homogeneous system of linear equation has only trivial solution if rank of A = number of variables.

Rule – II: Homogeneous system of linear equation has non-trivial solution if rank of A < number of variables.

4.10 Applications of Matrices in Real World

Matrices play a crucial role in solving real-world problems across various fields. In graphic design, they help manipulate images through transformations like scaling, rotation, and reflection. Data encryption and cryptography use matrices for secure communication by encoding and decoding messages. In seismic analysis, engineers use matrices to model and predict earthquake wave behavior. Geometric transformations, such as translation and dilation, rely on matrices to modify shapes in computer graphics. Additionally, social network analysis leverages matrices to represent and analyze relationships between individuals, identifying key influencers and connections in a network.

Transformation or Reflection Matrix is a mathematical tool that represents the reflection of a point or object across a mirror line in a coordinate plane. It's a matrix representation of a reflection transformation. In two dimensions, this typically means reflecting across the x-axis, y-axis or a line such as $y = x$.

To reflect a matrix over the x-axis, we have multiply it by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

To reflect a matrix over the y-axis, we have multiply it by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

To reflect a matrix over the line $y = x$, we have multiply it by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 16: A triangle has the vertices $A(2, 3)$, $B(-1, 4)$ and $C(3, -2)$. Find the vertices of the reflected triangle over the x-axis by using transformation matrix.

Solution: To reflect a point across a certain axis or line, we have multiply the point as a column vector by the corresponding transformation matrix.

Here, to reflect the given points over the x-axis, we use the transformation matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Write the points as column matrices

$$A=\begin{bmatrix}2\\3\end{bmatrix},\ B=\begin{bmatrix}-1\\4\end{bmatrix},\ C=\begin{bmatrix}3\\-2\end{bmatrix}$$

The vertex A' of the reflected image $=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}2+0\\0-3\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}=(2,-3)$

The vertex B' of the reflected image $=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}-1+0\\0-4\end{bmatrix}=\begin{bmatrix}-1\\-4\end{bmatrix}=(-1,-4)$

The vertex C' of the reflected image $=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}3-0\\0+2\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}=(3,2)$

Thus, the vertices of the reflected triangle are $A'(2,-3)$, $B'(-1,-4)$ and $C'(3,2)$.

Coding is the process of converting a message into a specific format using a code. A code is a system of symbols, words or signals used to represent other words or meanings. It's often used to hide the actual meaning of a message.

To decode a message, we multiply coded matrix by the inverse of the given matrix.

Example 17: Use matrix $A=\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}$ to encode the message: ATTACK, where letters A to Z are corresponding to the numbers 1 to 26.

Solution: Here

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Divide the letters of the message into groups of two.

AT TA CK

Assign the numbers to these letters and convert each pair of numbers into 2 × 1 matrices.

$$\begin{bmatrix}A\\T\end{bmatrix}=\begin{bmatrix}1\\20\end{bmatrix},\quad \begin{bmatrix}T\\A\end{bmatrix}=\begin{bmatrix}20\\1\end{bmatrix},\quad \begin{bmatrix}C\\K\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}$$

So, the message in 2 × 1 matrices is $\begin{bmatrix}1\\20\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}\begin{bmatrix}3\\11\end{bmatrix}$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A.

i.e., $\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}1\\20\end{bmatrix}=\begin{bmatrix}1 & + & 40\\3 & + & 20\end{bmatrix}=\begin{bmatrix}41\\23\end{bmatrix}$

$$\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}=\begin{bmatrix}20 & + & 2\\60 & + & 1\end{bmatrix}=\begin{bmatrix}22\\61\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}3\\11\end{bmatrix}=\begin{bmatrix}3 & + & 22\\9 & + & 11\end{bmatrix}=\begin{bmatrix}25\\20\end{bmatrix}$$

So, the desired coded message is $\begin{bmatrix}41\\23\end{bmatrix}\begin{bmatrix}22\\61\end{bmatrix}\begin{bmatrix}25\\20\end{bmatrix}$

EXERCISE 4.3

1. Find the inverses of the following matrices by using row operations:

(i) $\begin{bmatrix}1 & 2 & -3\\0 & -2 & 0\\-2 & 4 & 6\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2 & -1\\0 & -2 & 8\\1 & 0 & 2\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 6 & 2\\2 & 13 & 0\\0 & -1 & 1\end{bmatrix}$

2. Find the rank of the following matrices:

i) $\begin{bmatrix}1 & -1 & 3 & 1\\-2 & -6 & 1 & -1\\3 & 1 & 4 & -2\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -2 & 3\\2 & -4 & 6\\-1 & 0 & 2\\0 & 1 & -1\end{bmatrix}$ (iii) $\begin{bmatrix}3 & -1 & 3 & 0 & 1\\1 & 2 & -1 & -3 & -2\\2 & 3 & 4 & 2 & 1\\2 & 5 & -2 & -3 & 3\end{bmatrix}$

3. Solve the following systems of linear equations by Cramer's rule:

(i) $\left.\begin{aligned}2x+y-z&=1\\x-y+2z&=3\\3x+2y+z&=4\end{aligned}\right\}$ (ii) $\left.\begin{aligned}x_1+2x_2-3x_3&=0\\4x_1-x_2+x_3&=5\\-2x_1+3x_2+2x_3&=3\end{aligned}\right\}$ (iii) $\left.\begin{aligned}2x_1-x_2+x_3&=1\\x_1+2x_2+2x_3&=2\\x_1-2x_2-x_3&=1\end{aligned}\right\}$

4. Solve the following systems of linear equations by matrix inversion method:

(i) $\left.\begin{aligned}x-2y+z&=-1\\3x+y-2z&=4\\y-z&=1\end{aligned}\right\}$ (ii) $\left.\begin{aligned}2x_1+x_2+3x_3&=3\\x_1+3x_2-2x_3&=0\\-3x_1-x_2+2x_3&=4\end{aligned}\right\}$ (iii) $\left.\begin{aligned}x+y&=2\\2x-z&=1\\2y-3z&=-1\end{aligned}\right\}$

5. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms:

(i) $\left.\begin{aligned}x_1+2x_2-2x_3&=-1\\2x_1+3x_2+x_3&=1\\5x_1+4x_2-3x_3&=1\end{aligned}\right\}$ (ii) $\left.\begin{aligned}x+2y+z&=2\\2x+y+2z&=3\\2x+3y-z&=7\end{aligned}\right\}$ (iii) $\left.\begin{aligned}x_1+4x_2+x_3&=2\\2x_1+x_2-2x_3&=9\\3x_1+x_2-x_3&=12\end{aligned}\right\}$

6. Solve the following systems of homogeneous linear equations by using Gaussian elimination method:

(i) $\left.\begin{array}{r} x+4x-2z=0 \\ 2x+y+5z=0 \\ 5x+2y+8z=0 \end{array}\right\}$ (ii) $\left.\begin{array}{r} x_1+4x_2+2x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+2x_2-4x_3=0 \end{array}\right\}$ (iii) $\left.\begin{array}{r} x_1+2x_2-x_3=0 \\ x_1-x_2+5x_3=0 \\ 2x_1+x_2+4x_3=0 \end{array}\right\}$

7. A triangle has vertices at $A(4,1)$, $B(-2,5)$ and $C(0,-3)$. Find the vertices of the reflected triangle over the y-axis using a transformation matrix.

8. The point A is mapped to (30, 20, –5) by the scaling matrix $P=\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.

Find the coordinates of A.

[Hint: If A is mapped to A' by scaling matrix P, then $AP = A'$]

9. Find the equation of the image of the curve with equation $y = x^2$ under the transformation with associated matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

10. Use the matrix $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ to encode the message: KEEP IT UP, where letters A to Z are corresponding to the numbers 1 to 26.

11. Decode the message $\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}\begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}\begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix}$ that was encode using matrix $A=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, where the numbers 1 to 26 are corresponding to the letters A to Z.

Unit 5 Partial Fractions

INTRODUCTION

We have learnt in the previous classes how to add two or more rational fractions into a single rational fraction. For example,

(i) $$\frac{1}{x-1}+\frac{2}{x+2}=\frac{3x}{(x-1)(x+2)}$$

and (ii) $$\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2}=\frac{5x^2+5x-3}{(x+1)^2(x-2)}$$

In this unit we shall learn how to reverse the order in (i) and (ii) that is to express a single rational function as a sum of two or more single rational functions which are called **Partial Fractions**.

Expressing a rational function as a sum of partial fractions is called **Partial Fraction Resolution**. It is an extremely valuable tool in the study of calculus to decompose a complex rational function into a sum of simpler fractions.

An open sentence formed by using the sign of equality '=' is called an equation. The equations can be divided into the following two kinds:

Conditional equation: It is an equation in which two algebraic expressions are equal for particular values of the variable e.g.,

(a) $2x = 3$ is a conditional equation and it is true only if $x=\frac{3}{2}$.

(b) $x^2+x-6=0$ is a conditional equation and it is true for $x = 2, -3$ only.

Note:
For simplicity, a conditional equation is called an equation.

Identity: It is an equation which holds good for all values of the variable e.g.,

(a) $(a+b)x \equiv ax+bx$ is an **identity** and its two sides are equal for all values of x.

(b) $(x+3)(x+4) \equiv x^2+7x+12$ is also an identity which is true for all values of x.

For convenience, the symbol "=" shall be used both for equation and identity.

5.1 Rational Fraction

An expression of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x with real coefficients and $Q(x) \neq 0$, is called a rational fraction. A rational fraction is of two types.

5.1.1 Proper Rational Fraction

A rational function $\dfrac{P(x)}{Q(x)}$ is called a **Proper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example, $\dfrac{3}{x+1}$, $\dfrac{2x-5}{x^2+4}$ and $\dfrac{9x^2}{x^3-1}$ are proper rational fractions or proper fractions.

5.1.2 Improper Rational Fraction

A rational fraction $\dfrac{P(x)}{Q(x)}$ is called an **Improper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator.

For example, $\dfrac{x}{2x-3}$, $\dfrac{(x-2)(x+1)}{(x-1)(x+4)}$, $\dfrac{x^2-3}{3x+1}$ and $\dfrac{x^3-x^2+x+1}{x^2+5}$

are improper rational fractions or improper fractions.

Any improper rational fraction can be reduced by division to a mixed form, consisting of the sum of a polynomial and a proper rational fraction.

For example, $\dfrac{3x^2+1}{x-2}$ is an improper rational fraction. By long division we obtain $\dfrac{3x^2+1}{x-2} = 3x+6+\dfrac{13}{x-2}$

$$\begin{array}{r} 3x+6 \\ x-2\,\big)\,3x^2+1 \\ \pm 3x^2 \mp 6x \\ \hline 6x+1 \\ \pm 6x \quad 12 \\ \hline 13 \end{array}$$

i.e., an improper rational fraction has $\dfrac{3x^2-1}{x-2}$ been reduced to the sum of a polynomial $3x+6$ and a proper rational fraction $\dfrac{13}{x-2}$.

When a rational fraction is separated into partial fractions, the result is an identity; i.e., it is true for all values of the variable in the domain of identity.

The evaluation of the coefficients of the partial fractions is based on the following theorem:

> "*If two polynomials are equal for all values of the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal*".

For example,

If $px^3 + qx^2 - ax + b = 2x^3 - 3x^2 - 4x + 5$, $\forall\, x$ then $p = 2$, $q = -3$, $a = 4$ and $b = 5$.

5.1.3 Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

(i) The degree of $P(x)$ must be less than that of $Q(x)$. If not, divide and work with the remainder theorem.

(ii) Factor the denominator $Q(x)$ into its irreducible factor, write the rational fraction into partial fractions.

(iii) Multiply the identity with the denominator of left hand side.

(iv) Equate the coefficients of like terms (powers of x).

(v) Solve the resulting equations for the coefficients.

We now discuss the following cases of partial fractions resolution.

Case I: Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors:

The polynomial $Q(x)$ may be written as:

$$Q(x) = (x - a_1)(x - a_2) \ldots (x - a_n), \text{ where } a_1 \neq a_2 \neq \ldots \neq a_n$$

$$\therefore \quad \frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \ldots + \frac{A_n}{x - a_n} \text{ is an identity.}$$

Where $A_1, A_2, \ldots, A_n$ are numbers to be found.

The method is explained by the following examples:

Example 1: Resolve $\frac{7x + 25}{(x + 3)(x + 4)}$ into partial fractions.

Solution: Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

Multiplying both sides by $(x + 3)(x + 4)$, we get

$$7x + 25 = A(x + 4) + B(x + 3)$$

$$\Rightarrow \quad 7x + 25 = Ax + 4A + Bx + 3B$$

$$\Rightarrow \quad 7x + 25 = (A + B)x + 4A + 3B$$

this is an identity in x.

So, equating the coefficients of like powers of x we have

$$7 = A + B \quad \text{and} \quad 25 = 4A + 3B$$

Solving these equations, we get $\boxed{A = 4}$ and $\boxed{B = 3}$.

Hence, $\frac{7x+25}{(x+3)(x+4)} = \frac{4}{x+3} + \frac{3}{x+4}$.

Alternative method

Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

$$\Rightarrow \quad 7x + 25 = A(x + 4) + B(x + 3)$$

As two sides of the identity are equal for all values of x,

Let us put $x = -3$ and $x = -4$ in it.

For A, putting $x + 3 = 0$ i.e., $x = -3$ we get,

$$-21 + 25 = A(-3 + 4)$$

$$\Rightarrow \quad \boxed{A = 4}$$

For B, putting $x + 4 = 0$ i.e., $x = -4$ we get,

$$-28 + 25 = B(-4 + 3)$$

$$\Rightarrow \quad \boxed{B = 3}$$

Hence, $\frac{7x+25}{(x+3)(x+4)} = \frac{4}{x+3} + \frac{3}{x+4}$.

Example 2: Resolve $\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ into Partial Fractions.

Solution: The polynomial $x^2 - 5x + 6$ in the denominator can be factorized and its factors are $x - 3$ and $x - 2$.

$$\therefore \quad \frac{x^2-10x+13}{(x-1)(x^2-5x+6)} = \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

Suppose $\dfrac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$

$\Rightarrow \quad x^2 - 10x + 13 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

which is an identity in x.

For A, putting $x-3=0$ i.e., $x=1$, we get

$$(1)^2 - 10(1) + 13 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$\Rightarrow \quad 1 - 10 + 13 = A(-1)(-2) + B(0)(-2) + C(0)(-1)$$

$$4 = 2A$$

$$\therefore \quad \boxed{A = 2}$$

For B, putting $x-2=0$ i.e., $x=2$, we get

$$(2)^2 - 10(2) + 13 = A(0)(2-3) + B(2-1)(2-3) + C(2-1)(0)$$

$$\Rightarrow \quad 4 - 20 + 13 = B(1)(-1)$$

$$\Rightarrow \quad -3 = -B$$

$$\therefore \quad \boxed{B = 3}$$

For C, putting $x-3=0$ i.e., $x=3$, we get

$$(3)^2 - 10(3) + 13 = A(3-2)(0) + B(3-1)(0) + C(3-1)(3-2)$$

$$\Rightarrow \quad 9 - 30 + 13 = C(2)(1)$$

$$\Rightarrow \quad -8 = 2C$$

$$\therefore \quad \boxed{C = -4}$$

Hence partial fractions are: $\dfrac{2}{x-1}+\dfrac{3}{x-2}-\dfrac{4}{x-3}$

Example 3: Resolve $\dfrac{2x^3+x^2-x-3}{x(2x+3)(x-1)}$ into Partial Fractions.

Solution: $\therefore$ $\dfrac{2x^3+x^2-x-3}{x(2x+3)(x-1)}$ is an improper fraction so, first transform it into mixed form.

$$\begin{array}{r|l} & 1 \\ 2x^3+x^2-3x & 2x^3+x^2-x-3 \\ & \pm 2x^3 \pm x^2 \mp 3x \\ \hline & 2x-3 \end{array}$$

Denominator $= x(2x+3)(x-1) = 2x^3 + x^2 - 3x$

$\therefore$ Dividing $2x^3+x^2-x-3$ by $2x^3+x^2-3x$, we have

Quotient = 1 and Remainder = $2x-3$

$$\therefore \quad \frac{2x^3+x^2-x-3}{x(2x+3)(x-1)} = 1 + \frac{2x-3}{x(2x+3)(x-1)}$$

Suppose $$\frac{2x-3}{x(2x+3)(x-1)} = \frac{A}{x}+\frac{B}{2x+3}+\frac{C}{x-1}$$

$$\Rightarrow \quad 2x-3 = A(2x+3)(x-1)+B(x)(x-1)+C(x)(2x+3)$$

which is an identity in x.

For A, putting $x=0$ in the identity, we get $\boxed{A=1}$

For B, putting $2x+3=0 \Rightarrow x=-\frac{3}{2}$ in the identity, we get $\boxed{B=-\frac{8}{5}}$

For C, putting $x-1=0 \Rightarrow x=1$ in the identity, we get $\boxed{C=-\frac{1}{5}}$

Hence partial fractions are: $1+\frac{1}{x}-\frac{8}{5(2x+3)}-\frac{1}{5(x-1)}$

Case II: When $Q(x)$ has repeated linear factors:

If the polynomial $Q(x)$ has a repeated linear factors $(x-a)^n$, $n \geq 2$ and n is *a* positive integer, then $\frac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_n}{(x-a)^n}$$

where $A_1, A_2, \ldots, A_n$ are numbers to be found.

The method is explained by the following examples:

Example 4: Resolve $\frac{x^2+x-1}{(x+2)^3}$ into partial fractions.

Solution: Suppose $$\frac{x^2+x-1}{(x+2)^3} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$$

$$\Rightarrow \quad x^2+x-1 = A(x+2)^2+B(x+2)+C \qquad \text{(i)}$$

$$\Rightarrow \quad x^2+x-1 = A(x^2+4x+4)+B(x+2)+C \qquad \text{(ii)}$$

For C, putting $x+2=0$, i.e., $x=-2$ in (i), we get

$$(-2)^2+(-2)-1 = A(0)+B(0)+C$$

$$\Rightarrow \quad \boxed{1=C}$$

Equating the coefficients of x^2 and x in (ii), we get $\boxed{A=1}$

and $1 = 4A + B$

$\Rightarrow \quad 1 = 4 + B \quad \Rightarrow \quad \boxed{B=-3}$

Hence the partial fractions are: $\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}+\dfrac{1}{(x+2)^3}$

Example 5: Resolve $\dfrac{1}{(x+1)^2(x^2-1)}$ into partial fractions.

Solution: Here denominator $= (x+1)^2(x^2-1)$

$= (x+1)^2(x+1)(x-1) = (x+1)^3(x-1)$

$$\therefore \quad \frac{1}{(x+1)^2(x^2-1)}=\frac{1}{(x+1)^3(x-1)}$$

Suppose $$\frac{1}{(x-1)(x+1)^3}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$$

$\Rightarrow \quad 1 = A(x+1)^3 + B(x+1)^2(x-1) + C(x-1)(x+1) + D(x-1) \quad \ldots(i)$

$\Rightarrow \quad 1 = A(x^3+3x^2+3x+1) + B(x^3+x^2-x-1)+C(x^2-1) + D(x-1)$

$\Rightarrow \quad 1 = (A+B)x^3+(3A+B+C)x^2+(3A-B+D)x+(A-B-C-D) \quad \ldots(ii)$

For A, putting $x-1=0 \Rightarrow x=1$ in (i), we get

$$1 = A(2)^3 \quad \Rightarrow \quad \boxed{A=\frac{1}{8}}$$

For D, putting $x+1=0 \Rightarrow x=-1$ in (i), we get

$$1 = D(-1-1) \quad \Rightarrow \quad \boxed{D=-\frac{1}{2}}$$

Equating the coefficients of x^3 and x^2 in (ii), we get

$$0 = A+B \quad \Rightarrow \quad B=-A \quad \Rightarrow \quad \boxed{B=-\frac{1}{8}}$$

and $$0 = 3A+B+C \quad \Rightarrow \quad 0=\frac{3}{8}-\frac{1}{8}+C \quad \Rightarrow \quad \boxed{C=-\frac{1}{4}}$$

Hence the partial fractions are:

$$\frac{\frac{1}{8}}{x-1}+\frac{-\frac{1}{8}}{x+1}+\frac{-\frac{1}{4}}{(x+1)^2}+\frac{-\frac{1}{2}}{(x+1)^3}=\frac{1}{8(x-1)}-\frac{1}{8(x+1)}-\frac{1}{4(x+1)^2}-\frac{1}{2(x+1)^3}$$

EXERCISE 5.1

Resolve the following into partial fractions:

1. $\dfrac{1}{x^2-1}$
2. $\dfrac{(x^2+1)}{(x+1)(x-1)}$
3. $\dfrac{2x+1}{(x-1)(x+2)(x+3)}$
4. $\dfrac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$
5. $\dfrac{6x^3+5x2-7}{2x^2-x-1}$
6. $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$
7. $\dfrac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$ [Hint: Put $x^2=y$ to make factors of the denominator linear]
8. $\dfrac{2x^2-3x+4}{(x-1)^3}$
9. $\dfrac{5x^2-2x+3}{(x+2)^2}$
10. $\dfrac{4x}{(x+1)^2(x-1)}$
11. $\dfrac{2x^4}{(x-3)(x+2)^2}$

Case III: When $Q(x)$ contains non-repeated irreducible quadratic factors

Definition: A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, x^2+x+1 and x^2+3 are irreducible quadratic factors.

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factors then $\dfrac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form:

$$\frac{Ax+B}{ax^2+bx+c} \quad \text{where } A \text{ and } B \text{ are the numbers to be found.}$$

The method is explained by the following examples:

Example 6: Resolve $\dfrac{3x-11}{(x^2+1)(x+3)}$ into partial fractions.

Solution: Suppose $\dfrac{3x-11}{(x^2+1)(x+3)}=\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+3}$

$\Rightarrow \quad 3x-11=(Ax+B)(x+3)+C(x^2+1) \quad$ (i)

$\Rightarrow \quad 3x-11=(A+C)x^2+(3A+B)x+(3B+C) \quad$ (ii)

For C, putting $x+3=0 \quad \Rightarrow \quad x=-3$ in (i), we get

$-9-11=C(9+1) \quad \Rightarrow \quad \boxed{C=-2}$

Equating the coefficients of x^2 and x in (ii), we get

$$0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = 2}$$

and $\quad 3 = 3A + B \Rightarrow B = 3 - 3A \Rightarrow B = 3 - 6 \Rightarrow B = -3$

Hence, the partial fractions are: $\dfrac{2x-3}{x^2+1} - \dfrac{2}{x+3}$

Example 7: Resolve $\dfrac{4x^2+8x}{x^4+2x^2+9}$ into partial fractions.

Solution: Here, denominator $= x^4 + 2x^2 + 9 = (x^2 + 2x + 3)(x^2 - 2x + 3)$

$$\therefore \quad \frac{4x^2+8x}{x^4+2x^2+9} = \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Suppose $\quad \dfrac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = \dfrac{Ax+B}{x^2+2x+3} + \dfrac{Cx+D}{x^2-2x+3}$

$$\Rightarrow \quad 4x^2 + 8x = (Ax + B)(x^2 - 2x + 3) + (Cx + D)(x^2 + 2x + 3)$$

$$\Rightarrow \quad 4x^2 + 8x = (A + C)x^3 + (-2A + B + 2C + D)x^2 + (3A - 2B + 3C + 2D)x + 3B + 3D \qquad \text{(i)}$$

which is an identity in x.

Equating the coefficients of x^3, x^2, x, x^0 in (i), we have

$$0 = A + C \qquad \text{(ii)}$$

$$4 = -2A + B + 2C + D \qquad \text{(iii)}$$

$$8 = 3A - 2B + 3C + 2D \qquad \text{(iv)}$$

$$0 = 3B + 3D \qquad \text{(v)}$$

Solving (ii), (iii), (iv) and (v), we get

$$\boxed{A = 1}, \boxed{B = 2}, \boxed{C = -1} \text{ and } \boxed{D = -2}$$

Hence the partial fractions are: $\dfrac{x+2}{x^2+2x+3} + \dfrac{-x-2}{x^2-2x+3}$

Case IV: When $Q(x)$ has repeated irreducible quadratic factors

If the polynomial $Q(x)$ contains a repeated irreducible quadratic factors $(ax^2 + bx + c)^n$, $n \geq 2$ and n is a positive integer, then $\dfrac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are numbers to be found. The method is explained through the following example:

Example 8: Resolve $\frac{4x^2}{(x^2+1)^2(x-1)}$ into partial fractions.

Solution: Let $\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1}$

$\Rightarrow 4x^2 = (Ax+B)(x^2+1)(x-1) + (Cx+D)(x-1) + E(x^2+1)^2$ (i)

$\Rightarrow 4x^2 = (A+E)x^4 + (-A+B)x^3 + (A-B+C+2E)x^2$
$+ (-A+B-C+D)x + (-B-D+E)$ (ii)

For E, putting $x-1=0 \Rightarrow x=1$ in (i), we get

$4 = E(1+1)^2 \Rightarrow \boxed{E=1}$

Equating the coefficients of x^4, x^3, x^2, x, in (ii), we get

$0 = A+E \Rightarrow A = -E \Rightarrow \boxed{A=-1}$

$0 = -A+B \Rightarrow B = A \Rightarrow \boxed{B=-1}$

$4 = A-B+C+2E$

$\Rightarrow C = 4-A+B-2E = 4+1-1-2 \Rightarrow \boxed{C=2}$

and $0 = -A+B-C+D$

$\Rightarrow D = A-B+C = -1+1+2 = 2 \Rightarrow \boxed{D=2}$

Hence partial fractions are: $\frac{-x-1}{x^2+1} + \frac{2x+2}{(x^2+1)^2} + \frac{1}{x-1}$

EXERCISE 5.2

Resolve into partial fractions:

1. $\frac{9x-7}{(x^2+1)(x+3)}$
2. $\frac{x^2+15}{(x^2+2x+5)(x-1)}$
3. $\frac{x^2+1}{x^3+1}$
4. $\frac{x^4}{1-x^4}$
5. $\frac{2x-5}{(x^2+2)^2(x-2)}$
6. $\frac{8x^2}{(x^2+1)^2(1-x^2)}$

Unit 6 Sequences and Series

INTRODUCTION

In this unit, students will learn to analyze and solve problems involving arithmetic, geometric, and harmonic sequences and series, including their real-world applications. Learners will identify various sequence types, compute finite and infinite sums, and utilize sigma notation. Additionally, they will explore practical scenarios such as motor vehicle leasing, investment planning, and financial calculations. This unit also emphasizes applying these concepts to diverse fields, including healthcare, finance, and traffic modeling. Finally, Students will be able to solve both theoretical and real-life problems using sequences and series effectively.

Let us observe the following pattern of numbers.

(i) 5, 11, 17, 23, … (ii) 6, 12, 24, 48, …

(iii) 4, 2, 0, –2, – 4, … (iv) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, ...$

In example (i), every number (except 5) is formed by adding 6 to the previous numbers. Hence a specific pattern is followed in the arrangement of these numbers. Similarly, in example (ii), every number is obtained by multiplying the previous number by 2. Similar cases are followed in example (iii) and (iv). When a set of numbers follows a pattern and there is a clear rule for finding next number in the pattern, then we have sequence as in above examples.

6.1 Sequence

A systematic arrangement of numbers according to a given rule is called a sequence. The numbers in a sequence are called its **terms.** We refer the first term of a sequence as a_1, second term as a_2 and so on. The n^{th} term of a sequence is denoted by a_n, which may also be referred to as the general term of the sequence, and the terms immediately preceding it are called the $(n-1)$st term, the $(n-2)$nd term and so on.

6.1.2 Finite and Infinite Sequences

1. A sequence which consists of a finite number of terms is called a finite sequence. For example, 2, 5, 8, 11, 14, 17, 20, 23 is a finite sequence of 8 terms.
2. A sequence which consists of an infinite number of terms is called an infinite sequence. For example, 3, 10, 17, 24, … is an infinite sequence, or more generally as 3, 10, 17, 24, … , $7n-4$, … to show how each term was generated.

Note: If a sequence is given, then we can find its n term and if the n term of a sequence is given then we can find the terms of the sequence.

Example 1: Find the first four terms of the sequences whose n terms are given.

(i) $a_n = 3n + 1$

Substituting $n = 1$, we have

$$a_1 = 3(1) + 1 = 4$$

Similarly, $a_2 = 3(2) + 1 = 7$

$$a_3 = 3(3) + 1 = 10$$

$$a_4 = 3(4) + 1 = 13$$

The first four terms of the sequence are 4, 7, 10, 13

(ii) $a_n = 3n^2 - 3$

Substituting $n = 1$, we have

$$a_1 = 3(1)^2 - 3 = 0$$

Similarly, $a_2 = 3(2)^2 - 3 = 9$

$$a_3 = 3(3)^2 - 3 = 24$$

$$a_4 = 3(4)^2 - 3 = 45$$

The first four terms of the sequence are 0, 9, 24, 45

Sequences of numbers which follow specific patterns are called progression. Depending on the pattern, the progression is classified as follows.

(i) Arithmetic progression (ii) Geometric progression

(iii) Harmonic progression

1. Find the next four terms of each sequence.

   (i) 12, 16, 20, … (ii) 3, 1, –1, …

2. Write down the first three terms of each sequence.

   (i) $a_n = 3n + 5$ (ii) $a_{n+1} = 4a_n - 7$ and $a_1 = 3$

   (iii) $a_n = (n - 3)(n + 1)$ (iv) $a_1 = -1,\ a_{n+1} = \dfrac{3}{a_n + 2}$

   (v) $a_n = 8 - \dfrac{20}{3 + n}$ (vi) $a_1 = 1,\ a_{n+1} = (3a_n + 2)^2$

   (vii) $a_n = (-2n)^2$ (viii) $a_n = (-1)^n 7n^2$

3. An expression for the n^{th} triangular number is $\dfrac{n(n+1)}{2}$. Write down the 15th triangular number.

4. Write down the n^{th} term of each sequence.

(a) 7, 13, 19, 25, … (b) 7, 4, 1, –2, …

(c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$ (d) –15, – 4, 7, 18,…

5. The n^{th} term of the sequence 2, 0, –2, –4, … and the n^{th} term of the sequence –22, –20, –18, –16, …. are equal, find the value of n.

6.2 Arithmetic Progression or Arithmetic Sequence (A.P.)

A sequence $\{a_n\}$ is an arithmetic sequence or arithmetic progression (A.P.), if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$. The difference $a_n - a_{n-1}$ $(n > 1)$ i.e., the difference of two consecutive terms of an A.P., is called the **common difference** and is usually denoted by d.

Thus, an arithmetic progression is a sequence in which each term after the first is found by adding a constant to the previous term. This constant is called common difference of the arithmetic progression.

Note:
If $a_1, a_2, a_3, ... a_n, ...$ are in A.P., then $d = a_2 - a_1 = a_3 - a_2 = ...$ where a_n is n^{th} term of the A.P.

For example: Following sequences are A.P.

(i) 1, 3, 5, 7, … (common difference is 2)

(ii) 54, 51, 48, … (common difference is –3)

0An arithmetic progression with n terms can be written as:

$$a_1, a_1 + d, a_1 + 2d, ..., [a_1 + (n-1)d]$$

The n^{th} term of an arithmetic progression can be written as:

$$a_n = a_1 + (n-1)d$$

Note:

(i) $1^{st}, 2^{nd}, 3^{rd}$ and n^{th} terms of an A.P. are denoted by a_1, a_2, a_3 and a_n respectively.

(ii) n^{th} term from the end of an *A.P.* is $(m - n + 1)^{th}$ term where 'm' denotes the total number of terms of an A.P.

(iii) Three numbers a, b, c are in *A.P.* if and only if $2b = a + c$.

(iv) Any term (except first and last) in an A.P. is equal to half of the sum of two terms equidistant from it.

(v) If the term a_1 is unknown or not given, the nth term can be written as $a_n = a_m + (n - m)d$ (the subscript of the given term and coefficient of d sum to n)

The middle term of an A.P. depends upon the number of terms, e.g.,

(i) 1, 3, 5, 7, 9, 11 is an A.P. with $n = 6$

(ii) 1, 3, 5, 7, 9, 11, 13 is an A.P. with $n = 7$

i.e., If the total number of terms of an A.P. is even, then there are two middle terms i.e., $\left(\frac{n}{2}\right)th$ and $\left(\frac{n}{2}+1\right)th$ where n represent the number of terms. In example (i) 5, 7 are two middle terms.

If the total number of terms of an $A.P.$ is odd, then there is only one middle term i.e., $\left(\frac{n+1}{2}\right)th$ term. In example (ii) 7 is the only middle term.

6.2.1 Selection of terms in *A.P.*

(i) Three consecutive terms of an A.P. can be chosen as $a-d, a, a+d$ or $a, a+d$, or $a+2d$

(ii) Four consecutive term of an A.P. may be written like $a-3d, a-d, a+d, a+3d$ or $a, a+d, a+2d, a+3d$.

(iii) Last four consecutive terms if ℓ is the last term can be written as below:

$$\ell-3d, \ell-2d, \ell-d, \ell$$

If each term of an $A.P.$ is increased or decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an A.P. i.e., if $a_1, a_2, a_3, \ldots a_n$ are in A.P., then

(i) $a_1 \pm k, a_2 \pm k, \ldots, a_n \pm k, \ldots$ are also in A.P. with common difference 'd'.

(ii) $ka_1, ka_2, \ldots, ka_n, \ldots$ are in A.P. with common difference 'kd'.

(iii) $\frac{a_1}{k}, \frac{a_2}{k}, \ldots, \frac{a_n}{k}$ ... A.P are in A.P. common difference $\frac{d}{k}$.

(iv) Term by term addition or subtraction of two arithmetic progressions is also an A.P. i.e.,

If $a_1, a_2, a_3, \ldots a_n, \ldots$ and $b_1, b_2, b_3, \ldots b_n, \ldots$ are in A.P., then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also in A.P.

Example 2: Find the general term and the eleventh term of the A.P. whose first term and the common difference are 2 and –3 respectively. Also write its first four terms.

Solution: Here, $a_1 = 2, d = -3$

We know that $a_n = a_1 + (n-1)d$

So $a_n = 2 + (n-1)(-3) = 2 - 3n + 3$

or $a_n = 5 - 3n$ (i)

Thus, the general term of the $A.P.$ is $5 - 3n$

Putting $n = 11$ in (i), we have

$$a_{11} = 5 - 3(11)$$
$$= 5 - 33 = -28$$

We can find a_2, a_3, a_4 by putting $n = 2, 3, 4$ in (i), that is,

$$a_2 = 5 - 3(2) = -1$$
$$a_3 = 5 - 3(3) = -4$$
$$a_4 = 5 - 3(4) = -7$$

Hence, the first four terms of the sequence are: 2, –1, –4, –7.

Example 3: If the 5th term of an *A.P.* is 13 and 17th term is 49, find a_n and a_{13}.

Solution: Given that $a_5 = 13$ and $a_{17} = 49$

Putting $n = 5$ in $a_n = a_1 + (n-1)d$, we have $a_5 = a_1 + (5-1)d$

$$a_5 = a_1 + 4d$$

or $\quad 13 = a_1 + 4d \qquad$ ...(i)

Also $\quad a_{17} = a_1 + (17-1)d$

or $\quad 49 = a_1 + 16d$

or $\quad 49 = (a_1 + 4d) + 12d$

or $\quad 49 = 13 + 12d \quad$ by (i)

$\Rightarrow \quad 12d = 36 \quad \Rightarrow \quad d = 3$

From (i), $a_1 = 13 - 4d = 13 - 4(3) = 1$

Thus $\quad a_{13} = 1 + (13-1)3 = 37$ and

$$a_n = 1 + (n-1)3 = 3n - 2$$

Example 4: Find the number of terms in the *A.P.* ; if $a_1 = 3, d = 7$ and $a_n = 59$

Solution: Using $a_n = a_1 + (n-1)d$, we have

$$59 = 3 + (n-1) \times 7 \qquad (\because a_n = 59, a_1 = 3 \text{ and } d = 7)$$

or $\quad 56 = (n-1) \times 7 \Rightarrow (n-1) = 8 \Rightarrow n = 9$

Thus, the terms in the *A.P.* are 9.

Example 5: If $a_{n-2} = 3n - 11$ find the n^{th} term of the sequence.

Solution: Replacing n by $n + 2$, we have

$$a_{n+2-2} = 3(n+2) - 11$$
$$a_n = 3n + 6 - 11$$
$$a_n = 3n - 5$$

EXERCISE 6.2

1. Find the common difference and write the next two terms of each arithmetic sequence.
   (i) 9, 16, 23, … (ii) $5,\ 5+\sqrt{2},\ 5+2\sqrt{2},...$
2. Write the first three terms of each arithmetic sequence, with given information.
   (i) $a_1 = 2, d = 13$ (ii) $a_1 = 12, d = -13$
3. Find a_{n+1} and a_{2n} if $a_n = 4 + 3n$
4. Find the indicated term of each of the following arithmetic sequence.
   (i) $a_1 = 3, d = 7, a_{14} = 14$ (ii) $8, 3, -2, \ldots, a_{12}$
5. The 18^{th} term of a sequence is 367. The 30^{th} term of the sequence is 499. How many term of this sequence are less than 1000?
6. Is 301 a term of the A.P. of the 5, 11, 17, …?
7. If $2x$, $x + 8$, $3x + 1$ are in A.P., then find the value of x.
8. Which term of the A.P., 3, 8, 18, … is 123.
9. Which term of the A.P., 30, 29.5, 29, 28.5, … is the first negative term.
10. The 7^{th} term and 21^{st} terms of an A.P., are 37 and 107 respectively. Find the A.P. and its 100^{th} term.
11. If $\frac{1}{a-c}, \frac{1}{b-c}, \frac{1}{b-a}$ are in A.P., the show that $\frac{a-b}{a-c} = \frac{a-c}{b-a}$.
12. How many numbers of three digits are divisible by 7?
13. Find the 8^{th} term from the end of the A.P., 8, 11, 14, …, 185.
14. Find the n^{th} term of the progression $\left(\frac{3}{7}\right)^{10}, \left(\frac{10}{7}\right)^{10}, \left(\frac{17}{7}\right)^{10}, \cdots$. Is the progression an A.P.? Is it infinite?
15. If the arithmetic progression 3, 10, 17, … and 63, 65, 67, … are such that their n^{th} terms are equal, then find the value of n.
16. If the p^{th} term of an A.P. is q and the q^{th} term is p, prove that its nth term is $(p + q - n)$.
17. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$.
18. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in *A.P.*, show that the common difference is $\frac{a-c}{2ac}$.
19. If a_k and a_m denotes two different terms of an A.P., show that its n^{th} term is $a_k + (n-k)\left(\frac{a_k - a_m}{k-m}\right)$.

20. If $a_1, a_2, a_3, ..., a_n$ are positive and in A.P., prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

21. If the roots of the equation $(b-c)x^2+(c-a)x+(a-b)=0$ are equal. Show that a, b, c are in A.P.
22. If the sides of a right-angled triangle are in A.P., find the ratio of its sides.
23. If the n^{th} term of a progression is a linear expression in n, then prove that this progression is an A.P.

6.3 Arithmetic Mean (A.M.)

A number A is said to be the A.M. between the two numbers a and b if a, A, b are in A.P. If d is the common difference of this A.P., then $A - a = d$ and $b - A = d$.

Thus $\quad A - a = b - A$

or $\quad 2A = a + b$

$\Rightarrow \quad A = \dfrac{a+b}{2}$

Note: If $A_1, A_2, A_3, \ldots, A_n$ are said to be n A.Ms. between two numbers a and b, then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P.

Example 6: Find three A.Ms. between $\sqrt{2}$ and $3\sqrt{2}$.

Solution: Let A_1, A_2, A_3 be three A.Ms. between $\sqrt{2}$ and $3\sqrt{2}$. Then,

$$\sqrt{2}, A_1, A_2, A_3, 3\sqrt{2} \text{ are in } A.P.$$

Here $a_1=\sqrt{2}$, $a_5=3\sqrt{2}$ using $a_5=a_1+(5-1)d$ or $3\sqrt{2}=\sqrt{2}+4d$

$$\Rightarrow \quad d=\frac{2\sqrt{2}}{4}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

Now $\quad A_1=a_1+d=\sqrt{2}+\dfrac{1}{\sqrt{2}}=\dfrac{2+1}{\sqrt{2}}=\dfrac{3}{\sqrt{2}}$

$$A_2=A_1+d=\frac{3}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

$$A_3=A_2+d=2\sqrt{2}+\frac{1}{\sqrt{2}}=\frac{4+1}{\sqrt{2}}=\frac{5}{\sqrt{2}}$$

Therefore, $\dfrac{3}{\sqrt{2}}$, $2\sqrt{2}$, $\dfrac{5}{\sqrt{2}}$ are the three A.Ms. between $\sqrt{2}$ and $3\sqrt{2}$.

EXERCISE 6.3

1. Find A.M. between the given numbers

   (i) $2+\sqrt{3}i,\ 2-\sqrt{3}i$ (ii) $(a+b)^2,\ (a-b)^2$

2. If 6, 11, 16 are three A.Ms. between a and b, find a and b.

3. Insert five A.Ms. between $\sqrt{2}$ and $\frac{15}{\sqrt{2}}$.

4. The A.M. of two numbers is 7 and their product is 45. Find the numbers.

5. If n arithmetic means are inserted between a and b, prove that $d=\frac{b-a}{n+1}$, where d is the common difference.

6. If A is the A.M. between a and b, prove that $(a-A)^2+(A-b)^2=\frac{1}{2}(a-b)^2$.

7. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the A.M. between a and b, where $a \neq b$.

6.4 Series

The sum of the terms of a sequence is called the series of the corresponding sequence.
For example, $1+2+3+\cdots+n$ is a finite series of first n natural numbers.
The sum of first n terms of series is denoted by S_n.

We write, $S_n = a_1 + a_2 + \cdots + a_n$.

Here,
$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$.
$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is known as n^{th} partial sum.

The sum of the terms of an arithmetic sequence is called an arithmetic series.
To develop a formula for the sum of any arithmetic series, consider

$$S_n = a_1 + (a_1+d) + (a_1+2d) + \cdots + (\ell-2d) + (\ell-d) + \ell \quad (\text{where } a_n = \ell)$$
$$S_n = \ell + (\ell-d) + (\ell+2d) + \cdots + (a_1+2d) + (a_1+d) + a_1$$

Thus, $2S_n = (a_1+\ell)+(a_1+\ell)+(a_1+\ell)+\cdots+(a_1+\ell)+(a_1+\ell)+(a_1+\ell)$
$= n(a_1+\ell)$ [We have n terms of $(a_1+\ell)$]

$$S_n = \frac{n}{2}(a_1+\ell)$$

But, $\ell = a_1 + (n-1)d$ (Substitute ℓ in S_n)

Thus, $S_n = \frac{n}{2}[a_1 + a_1 + (n-1)d] = \frac{n}{2}[2a_1 + (n-1)d]$

Example 7: Find the sum of the first 100 positive integers.

Solution: The series is 1 + 2 + 3 + … + 100. Since you can see that $a_1 = 1, a_n = 100$ and $d = 1$, you can use either sum formula for this arithmetic series.

Key Concept

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \text{ or } S_n = \frac{n}{2}(a_1 + a_n)$$

Method-1	**Method-2**
$S_n = \frac{n}{2}(a_1 + a_n)$	$S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$S_{100} = \frac{100}{2}(1+100)$	$S_{100} = \frac{100}{2}[2(1) + (100-1)1]$
$S_{100} = 50(101)$	$S_{100} = 50(101)$
$S_{100} = 5050$	$S_{100} = 5050$

Example 8: Find the 19th term and the partial sum of 19 terms of the arithmetic series:

$$2 + \frac{7}{2} + 5 + \frac{13}{2} + \cdots$$

Solution: Here, $a_1 = 2$ and $d = a_2 - a_1 = \frac{3}{2}$

Using $\quad a_n = a_1 + (n-1)d$

$$a_{19} = 2 + (19-1)\frac{3}{2}$$

$$= 2 + 18\left(\frac{3}{2}\right) = 2 + 27 = 29$$

Using $\quad S_n = \frac{n}{2}(a_1 + a_n)$

$$S_{19} = \frac{19}{2}(2+29) = \frac{19}{2}(31) = \frac{589}{2}$$

Example 9: Find the arithmetic series if its fifth term is 19 and $S_4 = a_9 + 1$.

Solution: Given that $a_5 = 19$, that is,

$$a_1 + 4d = 19 \qquad \text{(i)}$$

Using the other given condition, we have

$$S_4 = \frac{4}{2}[2a_1 + (4-1)d] = a_9 + 1$$

or $\quad 4a_1 + 6d = a_1 + 8d + 1$

$$3a_1 - 1 = 2d$$

Substituting $2d = 3a_1 - 1$ in (i), we have

$$a_1 + 2(3a_1 - 1) = 19$$

or $\quad 7a_1 = 21 \Rightarrow a_1 = 3$

From (i), we have,

$$4d = 19 - a_1 = 19 - 3 = 16$$

$\Rightarrow \quad d = 4$

Thus, the series is $3 + 7 + 11 + 15 + 19 + \cdots$

Example 10: How many terms of the series $-9 - 6 - 3 + 0 + \cdots$ amounts to 66?

Solution: Here, $a_1 = -9$ and $d = 3$ as $-6 - (-9) = 3$.

Let $S_n = 66$

Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$, we have

$$66 = \frac{n}{2}[2(-9) + (n-1)3]$$

or $\quad 132 = n[3n - 21] \quad \Rightarrow \quad 44 = n(n-7)$

or $\quad n^2 - 7n - 44 = 0$

$$\Rightarrow \quad n = \frac{7 \pm \sqrt{49 + 176}}{2}$$

$$= \frac{7 \pm \sqrt{225}}{2} = \frac{7 \pm 15}{2} \quad \Rightarrow \quad n = 11, -4$$

But n cannot be negative in this case, so $n = 11$, that is, the sum of eleven terms amount to 66.

Example 11: Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$ and $S_n = 741$.

Solution:

Step – I: Since we know a_1, a_n and S_n,

Use $S_n = \frac{n}{2}(a_1 + a_n)$ to find n.

$$741 = \frac{n}{2}(9 + 105)$$

$$741 = 57n$$

$$13 = n$$

Step – II: Find d.

$$a_n = a_1 + (n-1)d$$

$$105 = 9 + (13-1)d$$

$$96 = 12d$$

$$8 = d$$

Step – III: Use d to determine a_2 and a_3.

$$a_2 = 9 + 8 = 17, \qquad a_3 = 17 + 8 = 25$$

The first three terms are 9, 17 and 25.

EXERCISE 6.4

1. Sum the series:

   (i) $3+6+9+...+a_{20}$ (ii) $\frac{4}{\sqrt{5}}+\sqrt{5}+\frac{6}{\sqrt{5}}+...+a_n$

2. Find S_n for each arithmetic series:

   (i) $a_1 = 4, n = 25, a_n=100$ (ii) $a_1 = 40, n = 20, d= -3$

   (iii) $a_n = 52, n = 21, d= -4$

3. Find a_1 for arithmetic series: $d = 8, n = 19, S_n = 1786$

4. How many terms of the series: 96 + 93 + 90 + … amount to 1071.

5. If the three sides of a right-angled triangle of perimeter equal to 36cm are in A.P. find them.

6. Sum the series

   (i) $3 + 5 - 7 + 9 + 11 - 13 + 15 + 17 - 19 + \dots$ to $3n$ terms.

   (ii) $1 + 4 - 7 + 10 + 13 - 16 + 19 + 22 - 25 + \dots$ to $3n$ terms.

7. Find the sum of 20 terms of the series whose r^{th} term is $3r + 1$.

8. The 5^{th} and 9^{th} term of an $A.P.$ are 11 and 17 respectively. Find the sum of 20 terms.

9. Obtain the sum of all integers in the first 1000 positive integers which are neither divisible by 5 nor by 2.

10. The sum of 9 terms of an $A.P.$ is 171 and its eighth term is 31. Find the series.

11. The 5^{th} term of an arithmetic progression is 21 and the sum of first six terms is 90. Find the 18^{th} term.

12. The sum of three numbers in an $A.P.$ is 24 and their product is 440. Find the numbers.

13. The first four terms of an $A.P.$ are 2, 6, 10 and 14. Find the least number of terms needed so that the sum of the terms is greater than 2000.

14. Find four numbers in $A.P.$ whose sum is 32 and the sum of whose squares is 276.

15. Find the five numbers in $A.P.$ whose sum is 25 and the sum of whose squares is 135.

16. If $\frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ are in $A.P.$ then show that a^2, b^2, c^2 are in $A.P.$

17. The sum of the first four terms of an *A.P.* is 56. The sum of the last four terms is 112. If its first term is 11, then find number of terms.
18. The first term of an A.P. is a, the second term is b and the last term is c. show that the sum of A.P. is $\dfrac{(b+c-2a)(c+a)}{2(b-a)}$.
19. Show that the sum of n A.Ms. between a and b is n times the single A.M. between them.

6.5 Geometric Progression (G.P.)

A geometric progression or geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a nonzero constant r called common ratio.

Like arithmetic progression, we can label the terms of a geometric sequence as a_1, a_2, a_3 and so on, $a_1 \neq 0$. The n^{th} term is a_n and the previous term is a_{n-1}. So, $a_n = r(a_{n-1})$. Thus, $r = \dfrac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

6.5.1 Rule for *n*th term of a G.P.

Each term after the first term is an r multiple of its preceding term. Thus, we have,

$$a_2 = a_1 r = a_1 r^{2-1}$$
$$a_3 = a_2 r = (a_1 r)r = a_1 r^2 = a_1 r^{3-1}$$
$$a_4 = a_3 r = (a_1 r^2)r = a_1 r^3 = a_1 r^{4-1}$$
$$\vdots$$
$$a_n = a_1 r^{n-1}$$ which is the general term of a G.P.

6.5.2 Properties of G.P.

(i) If each term of a G.P. is multiplied or divided by the same non-zero number, then the resulting sequence is also a G.P. i.e., if $g_1, g_2, g_3, ..., g_n, ...$ are in *G.P.* and k is a non-zero number, then

(a) $kg_1, kg_2, kg_3, ..., kg_n, ...$ are in G.P.

(b) $\dfrac{g_1}{k}, \dfrac{g_2}{k}, \dfrac{g_3}{k}, ..., \dfrac{g_n}{k}, ...$ are also in G.P.

(ii) The reciprocals of the term of a G.P. also form a G.P. i.e., if a, b, c are in G.P., then $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are also in G.P.

(iii) If each term of a G.P. be raised to the same power, the resulting numbers also form a G.P. i.e., if a, b, c are in G.P., then a^n, b^n, c^n are also in G.P.

(iv) Three numbers a, b, c are in G.P. if and only if $b^2 = ac$.

(v) If the set of positive numbers $a_1, a_2, a_3, ..., a_n, ...$ are in G.P., then $\log a_1$, $\ell og\ a_2$, $\ell og\ a_3$, ..., $\ell og\ a_n$, ... are also in A.P. and vice-versa.

(vi) Term by term multiplication or division of two G.Ps. are also in G.P. i.e., if $a_1, a_2, a_3 \ldots, a_n$, and $b_1, b_2, b_3, \ldots, b_n$, are in $G.P.$ then $a_1b_1, a_2b_2, a_3b_3, \ldots$, and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are also in $G.P.$

Example 12: Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.

Solution: Here, $a_1 = -3$, $r = -2$, $n = 8$

$$a_n = a_1 \cdot r^{n-1}$$
$$a_8 = (-3)\cdot(-2)^{8-1}$$
$$a_8 = (-3)\cdot(-128)$$
$$a_8 = 384$$

Example 13: Write an equation for the nth term of the geometric sequence 3, 12, 48, 192, …

Solution: Here $a_1 = 3$, $r = 4$

$$a_n = a_1 \cdot r^{n-1}$$
$$a_n = 3 \cdot 4^{n-1}$$

Example 14: Find the tenth term of a geometric sequence for which $a_4 = 108$ and $r = 3$.

Solution:

Step 1: Find the value of a_1.

Here, $n = 4$, $r = 3$, $a_4 = 108$

$$a_n = a_1 \cdot r^{n-1}$$
$$a_4 = a_1 \cdot 3^{4-1}$$
$$108 = 27a_1$$
$$4 = a_1$$

Step 2: Find a_{10}.

Here, $n = 10$, $a_1 = 4$, $r = 3$

$$a_n = a_1 \cdot r^{n-1}$$
$$a_{10} = 4 \cdot 3^{10-1}$$
$$a_{10} = 78{,}732$$

Example 15: Find the 5th term of the $G.P.$, 3, 6, 12, …

Solution: Here $a_1 = 3$, $a_2 = 6$, $a_3 = 12$, therefore, $r = \frac{a_2}{a_1} = \frac{6}{3} = 2$.

Using $a_n = a_1 r^{n-1}$ for $n = 5$, we have

$$a_5 = a_1 r^{5-1} = 3\cdot 2^{5-1} = 3\cdot 2^4 = 48$$

Example 16: Find a_n if $a_4 = \frac{8}{27}$ and $a_7 = \frac{-64}{729}$ of a *G.P.*

Solution: To find a_n we have to find a_1 and r.

Using $\quad a_n = a_1 r^{n-1} \qquad$ (i)

$$a_4 = a_1 r^{4-1} = a_1 r^3 \quad , \text{ so } \quad a_1 r^3 = \frac{8}{27} \qquad \text{(ii)}$$

and $\quad a_7 = a_1 r^{7-1} = a_1 r^6 \quad$, so $\quad a_1 r^6 = \frac{-64}{729} \qquad$ (iii)

Thus, $\quad \frac{a_7}{a_4} = \frac{\frac{-64}{729}}{\frac{8}{27}} = \frac{-8}{27} \qquad$ or $\quad r^3 = \left(\frac{-2}{3}\right)^3 \qquad \left(\because \frac{a_7}{a_4} = \frac{a_1 r^6}{a_1 r^3} = r^3\right)$

$\Rightarrow \qquad r = -\frac{2}{3} \qquad$ (taking only real value of r)

Put $r^3 = -\frac{8}{27}$ in (ii), to obtain a_1 that is,

$$a_1\left(-\frac{8}{27}\right) = \frac{8}{27} \quad \Rightarrow \quad a_1 = -1$$

Now putting $a_1 = -1$ and $r = \frac{-2}{3}$ in (i), we get,

$$a_n = (-1)\left(-\frac{2}{3}\right)^{n-1} = (-1)(-1)^{n-1}\cdot\left(\frac{2}{3}\right)^{n-1} = (-1)^n\left(\frac{2}{3}\right)^{n-1} \text{ for } n \geq 1.$$

EXERCISE 6.5

1. Find the 6th term of the *G.P.*: –6, –3, $\frac{-3}{2}$, …
2. Find the 8th term of the sequence, 3, 3^2, 3^3, …
3. The n^{th} terms of the sequences 1, 2, 4, 8, … and 256, 128, 64, … are equal. Find the value of n.
4. Find the first five terms of each sequence described:

   (i) $a_1 = 243,\ r = \frac{1}{3}$ (ii) $a_1 = 579,\ r = -\frac{1}{2}$

5. Find the 12^{th} term of $1 + i, 2i, -2 + 2i, \ldots$

6. If the 4^{th} and 9^{th} term of a G.P. are 54 and 13122 respectively. Find the G.P. Also find its general term.

7. If a, b, c, d are in G.P., prove that:
   (i) $a - b, b - c, c - d$ are in G.P.
   (ii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.
   (iii) $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are in G.P.

8. If $(p + q)^{th}$ term of a G.P. be m and $(p - q)^{th}$ term be n, then find the p^{th} term.

9. Find three consecutive numbers in G.P. whose sum is 26 and their product is 216.

10. The 3^{rd} term of a G.P. is the square of 1^{st} term. If the 2^{nd} term is 9 then find the 6^{th} term.

11. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. Show that the common ratio is $\pm\sqrt{\frac{a}{c}}$.

12. If the numbers 1, 4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the original numbers if their sum is 21.

13. If three consecutive numbers in A.P. are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 6.

15. If p^{th}, q^{th} terms of a G.P. are q and p respectively, show tha $(p + q)^{th}$ term is $\left(q^p \div p^q\right)^{\frac{1}{p-q}}$.

16. If $a, 2a + 2, 3a + 3, \ldots$ are in G.P., then find the fifth term.

6.6 Geometric Mean (G.M.)

A number G is said to be a geometric mean ($G.M.$) between two numbers a and b if a, G, b are in $G.P.$ Therefore,

$$\frac{G}{a} = \frac{b}{G}$$

$$\Rightarrow \quad G^2 = ab$$

$$\Rightarrow \quad G = \pm\sqrt{ab}$$

Note: $G_1, G_2, G_3, \ldots, G_n$ are said to be n G.Ms. between two numbers a and b if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P.

6.6.1 Relation Between A.M. and G.M.

If A and G are respectively A.M. and G.M. between two numbers a and b i.e.,

$A = \frac{a+b}{2}$ and $G = \sqrt{ab}$, then

(i) $A > G$ if $a \neq b$ (ii) $A = G$ if $a = b$

Example 17: Insert three *G.Ms* between 2 and $\frac{1}{2}$.

Solution: Let G_1, G_2, G_3 be three G.Ms. between 2 and $\frac{1}{2}$. Therefore

2, G_1, G_2, G_3, $\frac{1}{2}$ are in *G.P.* Here $a_1 = 2$, $a_5 = \frac{1}{2}$ and $n = 5$.

using $a_n = a_1 r^{n-1}$ we have

$$a_5 = a_1 r^{5-1} \quad \text{i.e.,} \quad a_5 = a_1 r^4 \qquad \text{(i)}$$

Now substituting the values of a_5 and a_1 in (i) we have

$$\frac{1}{2} = 2r^4 \quad \text{or} \quad r^4 = \frac{1}{4} \qquad \text{(ii)}$$

Taking square root of (ii), we get

$$r^2 = \pm\frac{1}{2}$$

We have, $r^2 = \frac{1}{2}$ or $r^2 = -\frac{1}{2} = \frac{i^2}{2}$ $(\because\ -1 = i^2)$

$\Rightarrow \quad r = \pm\frac{1}{\sqrt{2}}$ or $r = \pm\frac{1}{\sqrt{2}}i$

When $r = \frac{1}{\sqrt{2}}$, then $G_1 = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$, $G_2 = 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1$, $G_3 = 2\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$

When $r = \frac{-1}{\sqrt{2}}$, then $G_1 = 2\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}$, $G_2 = 2\left(\frac{-1}{\sqrt{2}}\right)^2 = 1$, $G_3 = 2\left(\frac{-1}{\sqrt{2}}\right)^3 = -\frac{1}{\sqrt{2}}$

When $r = \frac{i}{\sqrt{2}}$, then $G_1 = 2\left(\frac{i}{\sqrt{2}}\right) = \sqrt{2}i$, $G_2 = 2\left(\frac{i}{\sqrt{2}}\right)^2 = -1$, $G_3 = 2\left(\frac{i}{\sqrt{2}}\right)^3 = -\frac{i}{\sqrt{2}}$

When $r = \frac{-i}{\sqrt{2}}$, then $G_1 = 2\left(\frac{-i}{\sqrt{2}}\right) = -\sqrt{2}i$, $G_2 = 2\left(\frac{-i}{\sqrt{2}}\right)^2 = -1$, $G_3 = 2\left(\frac{-i}{\sqrt{2}}\right)^3 = \frac{i}{\sqrt{2}}$

Note: The real values of *r* are usually taken but here other cases are considered to widen the outlook of the students.

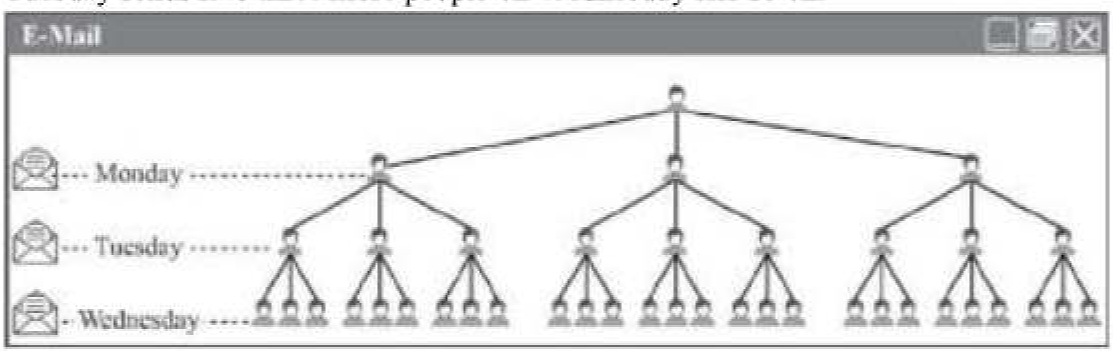
EXERCISE 6.6

1. Find G.M. between:
   (i) –2 and 8 (ii) –2*i* and 8*i* (iii) 6 and 9
2. Insert four real geometric means between 3 and 96.
3. If both *x* and *y* are positive distinct real numbers, show that the geometric mean between *x* and *y* is less than their arithmetic mean.

4. For what value of n, $\dfrac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the positive geometric mean between a and b?
5. The $A.M.$ of two positive integral numbers exceeds their (positive) $G.M.$ by 2 and their sum is 20, find the numbers.
6. The $A.M.$ between two numbers is 5 and their (positive) $G.M.$ is 4. Find the numbers.
7. The arithmetic mean between two positive numbers a and b is double their geometric mean. Prove that $a:b=2+\sqrt{3}:2-\sqrt{3}$
8. If one geometric mean G and two arithmetic means p and q be inserted between two positive numbers, show that $G^2=(2p-q)(2q-p)$

6.7 Geometric Series

Suppose you e-mail an Islamic quote to three friends on Monday. Each of those friends send it to three of their friends on Tuesday. Each person who receives the quote on Tuesday sends it to three more people on Wednesday and so on.



Notice that every day, the number of people who read your Islamic quote is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the quote is 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 or 3280. The numbers 1, 3, 9, 27, 81, 243, 729 and 2187 form a geometric sequence in which $a_1=1$ and $r=3$. The indicated sum of the numbers in the sequence, 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 is called a geometric series.

The sum of a geometric progression can be written as: $S_n=\dfrac{a_1(1-r^n)}{1-r},\ r\neq 1$

To develop a formula for the sum of any geometric series, consider

$$S_n=a_1+a_1r+a_1r^2+\cdots+a_1r^{n-3}+a_1r^{n-2}+a_1r^{n-1} \qquad \text{(i)}$$

$$rS_n=\quad a_1r+a_1r^2+\cdots+a_1r^{n-3}+a_1r^{n-2}+a_1r^{n-1}+a_1r^n \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$S_n - rS_n = a_1 - ar^n$$

$$S_n(1-r) = a_1(1-r^n)$$

$$S_n = \frac{a_1(1-r^n)}{1-r}, \; r \neq 1$$

Note: If $r = 1$, then $S_n = na_1$

Example 18: Find the sum of n terms of the geometric series if $a_n = (-3)\left(\frac{2}{5}\right)^n$.

Solution: We can write $(-3)\left(\frac{2}{5}\right)^n$ as:

$$-3\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{n-1} = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}, \text{ that is,}$$

$$a_n = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$$

Identifying $\left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$ with a_1r^{n-1}, we have $a_1 = -\frac{6}{5}$ and $r = \frac{2}{5}$

Thus, $$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{1-\frac{2}{5}}$$

$$= \left(-\frac{6}{5}\right)\left(\frac{5}{3}\right)\left[1-\left(\frac{2}{5}\right)^n\right] = (-2)\left[1-\left(\frac{2}{5}\right)^n\right]$$

EXERCISE 6.7

1. Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \cdots$.
2. The 3rd term of a $G.P.$ is 16 and the 6th term is –128. Find the first term and the sum of the first seven terms.
3. Sum to n terms the series:

   (i) $0.2 + 0.22 + 0.222 + \cdots$ (ii) $3 + 33 + 333 + \cdots$
4. Sum to n terms the series

   (i) $1 + (a+b) + (a^2 + ab + b^2) + (a^3 + a^2b + ab^2 + b^3) + \cdots$

   (ii) $r + (1+k)r^2 + (1+k+k^2)r^3 + \cdots$

5. Sum the series $2 + (1 - i) + \left(\frac{1}{i}\right) + \cdots$ to 8 terms.

6. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$, where r is the common ratio of G.P.

6.8 Arithmetico-Geometric Progression (A.G.P.)

Suppose $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an A.P., and $b_1, b_2, b_3, \ldots, b_n, \ldots$ is a G.P. then the sequence formed by multiplying the corresponding terms of A.P. and G.P., that is, a_1b_1, $a_2b_2, a_3b_3, \ldots, a_nb_n, \ldots$ is said to be an arithmetico-geometric sequence.

Consider an A.P., $a, a + d, a + 2d, \ldots, \{a + (n - 1)d\}$ and a G.P., $b, br, br^2, \ldots, br^{n-1}$ where $r \neq 1$.

Multiplying the corresponding terms of A.P. and G.P., we get an arithmetico-geometric sequence

$$ab, (a + d)br, (a + 2d)br^2, \ldots, \{a + (n - 1)d\}br^{n-1}$$

The n^{th} term of arithmetico-geometric sequence is product of n^{th} term of A.P. and n^{th} term of G.P. Thus, n^{th} term of such sequence has the form

$$\{a + (n - 1)d\}br^{n-1}$$

6.8.1 Arithmetico-Geometric Series

Sum of the terms of arithmetico-geometric sequence is called arithmetico-geometric series. Thus, arithmetico-geometric series has the form

$$ab + (a + d)br + (a + 2d)br^2 + \cdots + \{a + (n - 1)d\}br^{n-1}$$

Sum of *n*th Terms of Arithmetico-Geometric Series

Let $S_n = ab + (a + d)br + (a + 2d)br^2 + \cdots, + [a + (n - 1)d]br^{n-1}$ (i)

Then $rS_n = abr + (a + d)br^2 + \cdots + [a + (n - 2)d]br^{n-1} + [a + (n - 1)d]br^n$ (ii)

Subtracting (ii) from (i), we get

$$(1 - r)S_n = ab + [dbr + dbr^2 + \cdots + dbr^{n-1}] - [a + (n - 1)d]br^n$$

$$= ab + \frac{dbr(1-r^{n-1})}{1-r} - [a + (n - 1)d]br^n$$

$$= ab + \frac{dbr}{1-r} - \frac{dbr^n}{1-r} - [a + (n - 1)d]br^n$$

$$S_n = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2} - \frac{dbr^n}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r} \quad \text{(iii)}$$

which is the required sum of the n terms of arithmetico-geometric series.

6.8.2 Sum to Infinity of Arithmetico-Geometric Series

If $|r| < 1$, then $r^n \to 0$ and $nr^n \to 0$ as $n \to \infty$

Therefore, (iii) reduces to $S_\infty = \dfrac{ab}{1-r} + \dfrac{dbr}{(1-r)^2}$

which is the required sum to infinity of arithmetico-geometric series.

Example 19: Sum the series upto n terms: $2\cdot1 + 3\cdot2 + 4\cdot4 + 5\cdot8 + \ldots$

Solution: Let $S_n = 2\cdot1 + 3\cdot2 + 4\cdot2^2 + 5\cdot2^3 + \ldots$ to n terms

nth term of the A.P., 2, 3, 4, 5, … is $a_1 + (n-1)d = 2 + (n-1)(1)$

$$= 2 + n - 1$$
$$= n + 1$$

nth term of the G.P., 1, 2, 2^2, 2^3, … is $a_1 r^{n-1} = 1.2^{n-1} = 2^{n-1}$

So, $S_n = 2\cdot1 + 3\cdot2 + 4\cdot2^2 + 5\cdot2^3 + \ldots + (n+1)2^{n-1}$ (i)

Multiplying both sides by common ratio of G.P., we get

$$2S_n = 2\cdot2 + 3\cdot2^2 + 4\cdot2^3 + 5\cdot2^4 + \ldots + (n)2^{n-1} + (n+1)2^n \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$S_n - 2S_n = 2\cdot1 + (3-2)\cdot2 + (4-3)\cdot2^2 + (5-4)\cdot2^3 + \ldots + (n+1-n)2^{n-1} - (n+1)2^n$$
$$-S_n = 2\cdot1 + 1\cdot2 + 1\cdot2^2 + 1\cdot2^3 + \ldots + 1\cdot2^{n-1} - (n+1)2^n$$
$$-S_n = 2 + \{2 + 2^2 + 2^3 + \ldots + 2^{n-1}\} - (n+1)2^n$$
$$-S_n = 2 + \frac{2(2^{n-1}-1)}{2-1} - (n+1)\cdot2^n$$
$$-S_n = 2 + 2^n - 2 - n\cdot2^n - 2^n$$
$$-S_n = -n\cdot2^n$$
$$S_n = n\cdot2^n$$

Example 20: Sum the series upto n terms: $3\cdot1 + 4\cdot2 + 5\cdot2^2 + 6\cdot2^3 + \ldots$

Solution: Let $S_n = 3\cdot1 + 4\cdot2 + 5\cdot2^2 + 6\cdot2^3, \ldots$

nth term of the A.P., 3, 4, 5, 6, … is $a_1 + (n-1)d = 3 + (n-1)(1)$

$$= 3 + n - 1$$
$$= n + 2$$

nth term of the G.P., 1, 2, 2^2, 2^3, … is $a_1 r^{n-1} = 1(2)^{n-1} = 2^{n-1}$

So, $S_n = 3\cdot1 + 4\cdot2 + 5\cdot2^2 + 6\cdot2^3 + \ldots + (n+1)2^{n-1}$ (i)

Multiplying both sides by common ratio of G.P., we get

$$2S_n = 3\cdot2 + 4\cdot2^2 + 5\cdot2^3 + \ldots + (n+1)2^{n-1} + (n+1)2^n \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$S_n - 2S_n = 3\cdot 2 + (4-3)\cdot 2 + (5-4)\cdot 2^2 + (6-5)\cdot 2^3 + \ldots + (n+2-n-1)2^{n-1} - (n+2)2^n$$

$$-S_n = 3\cdot 1 + 1\cdot 2 + 1\cdot 2^2 + 1\cdot 2^3 + \ldots + 1\cdot 2^{n-1} - (n+2)2^n$$

$$-S_n = 3 + \{2 + 2^2 + .2^3 + \ldots + 2^{n-1}\} - (n+1)2^n$$

$$-S_n = 3 + \frac{2(2^{n-1}-1)}{2-1} - (n+2)2^n$$

$$-S_n = 3 + 2^n - 2 - n.2^n - 2.2^n$$

$$-S_n = 1 + 2^n - n\cdot 2^n - 2\cdot 2^n = 1 + (1-n-2)2^n$$

$$-S_n = 1 + (-n-1)2^n$$

$$S_n = -1 + (n+1)2^n$$

Example 21: Sum the series upto n terms: $2+\frac{4}{3}+\frac{6}{9}+\frac{8}{27}+\ldots$

Solution: Let $S_n = 2+\frac{4}{3}+\frac{6}{9}+\frac{8}{27}+\ldots$ to n terms

nth term of the A.P., 2, 4, 6, 8, … is $= 2 + (n-1)(2)$

$= 2 + 2n - 2 = 2n$

nth term of the G.P., $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $(1)\left(\frac{1}{3}\right)^{n-1} = \frac{1}{3^{n-1}}$

So, $$S_n = 2+\frac{4}{3}+\frac{6}{9}+\frac{8}{27}+\ldots+\frac{2n}{3^{n-1}} \qquad \text{(i)}$$

$$\frac{1}{3}S_n = \frac{2}{3}+\frac{4}{9}+\frac{6}{27}+\ldots+\frac{2n-2}{3^{n-1}}+\frac{2n}{3^n} \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$\left(1-\frac{1}{3}\right)S_n = 2+\frac{4-2}{3}+\frac{6-4}{9}+\frac{8-6}{27}+\ldots+\frac{2n-2n+2}{3^{n-1}}-\frac{2n}{3^n}$$

$$\frac{2}{3}S_n = 2+\left[\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots+\frac{2}{3^{n-1}}\right]-\frac{2n}{3^n}$$

$$\frac{2}{3}S_n = 2+\left[\frac{\frac{2}{3}\left\{1-\left(\frac{1}{3}\right)^{n-1}\right\}}{1-\frac{1}{3}}\right]-\frac{2n}{3^n}$$

$$= 2 + \frac{\frac{2}{3}\left\{1-\left(\frac{1}{3}\right)^{n-1}\right\}}{\frac{2}{3}} - \frac{2n}{3^n}$$

$$= 2 + 1 - \left(\frac{1}{3}\right)^{n-1} - 2n\left(\frac{1}{3}\right)^n$$

$$\frac{2}{3}S_n = 3 - \left(\frac{1}{3}\right)^{n-1} - 2n\left(\frac{1}{3}\right)^n$$

$$S_n = \frac{9}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^{n-1} - 3n\left(\frac{1}{3}\right)^n$$

Example 22: Find the sum to n terms of the series: $1 + 2x + 3x^2 + 4x^3 + \ldots$ where $x \neq 1$. If $|x| < 1$, sum the series to infinity.

Solution: Let $S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$ …(i)

$\therefore \quad xS_n = x + 2x^2 + 3x^3 + \ldots + (n-1)x^{n-1} + nx^n$ …(ii)

Subtracting (ii) from (i), we get

$$(1-x)S_n = 1 + x + x^2 + x^3 + \ldots + x^{n-1} - nx^n$$

$$= \frac{1(1-x^n)}{1-x} - nx^n$$

$$= \frac{1-x^n - n(1-x)x^n}{1-x}$$

$$= \frac{1-x^n - nx^n + nx^{n+1}}{1-x}$$

$$(1-x)S_n = \frac{1-(n+1)x^n + nx^{n+1}}{1-x}$$

$$S_n = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$

$$\therefore \quad S_\infty = \frac{1}{(1-x)^2}$$

EXERCISE 6.8

1. Find the 8th term of the arithmetico-geometric sequence, where the arithmetic part is 1, 4, 7, … and the geometric part is 5, 10, 20, …
2. Find the n^{th} term of the arithmetic-geometric sequence, where the arithmetic part is 3, 7, 11, …, and the geometric part is 2, 6,18, …
3. Consider the arithmetico-geometric sequence defined by arithmetic part: $a_{n+1} = 2n+5$ and geometric part: $b_{n-2} = \frac{1}{9}(-3)^n$. Find the n^{th} term and the sum of first three terms of the arithmetico-geometric sequence.
4. Sum to n terms the following series:
   (i) $1\cdot2 + 3\cdot4 + 5\cdot8 + 7\cdot16 + \dots$
   (ii) $2\cdot3 + 4\cdot3^2 + 6\cdot3^3 + 8\cdot3^4 + \dots$
   (iii) $2+\frac{5}{4}+\frac{8}{4^2}+\frac{11}{4^3}+\dots$
   (iv) $1+\frac{3}{5}+\frac{5}{5^2}+\frac{7}{5^3}+\dots$
   (v) $1+\frac{4}{3}+\frac{7}{9}+\frac{10}{27}+\dots$
5. Sum the following infinite series:
   (i) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\dots$
   (ii) $2+\frac{5}{3}+\frac{8}{9}+\frac{11}{27}+\dots$
6. Show that $2^{\frac{1}{2}}\cdot 4^{\frac{1}{4}}\cdot 8^{\frac{1}{8}}\cdot 16^{\frac{1}{16}}\cdots\infty = 4$
7. Show that $\sqrt{4}\cdot\sqrt[4]{16}\cdot\sqrt[8]{64}\cdot\sqrt[16]{256}\cdots\infty = 16$
8. Sum to n terms the series $2+4x+6x^2+8x^3+\cdots$ where $x \neq 1$
9. Find the sum to n terms of the series: $\frac{2n+1}{2n-1}+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\cdots$
10. Prove that: $1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+\cdots$ to n terms $= n^2$
11. Sum the series to n terms $2+5x+8x^2+11x^3+\cdots$ and deduce the sum to infinity if $|x|<1$.

6.9 Harmonic Progression (H.P.)

A sequence of numbers is called a Harmonic Sequence or Harmonic Progression if the reciprocals of its terms are in arithmetic progression. The sequence 1, $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ is a harmonic sequence since their reciprocals 1, 3, 5, 7 are in A.P.

Remember that the reciprocal of zero is not defined, so zero cannot be the term of a harmonic sequence.

The general form of a harmonic sequence is taken as:

$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots$ whose n^{th} term is $\frac{1}{a_1+(n-1)d}$

Example 23: Find the n^{th} and 8^{th} terms of H.P. : $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$

Solution: The reciprocals of the terms of the sequence,

$$\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots \quad \text{are } 2, 5, 8, \ldots$$

The numbers 2, 5, 8, … are in A.P., So

$$a_1 = 2 \text{ and } d = 5 - 2 = 3$$

Putting these values in $a_n = a_1 + (n-1)d$, we have

$$a_n = 2 + (n-1)3$$
$$= 3n - 1$$

Thus, the n^{th} term of the given sequence $= \frac{1}{a_n} = \frac{1}{3n-1}$ and substituting $n = 8$ in $\frac{1}{3n-1}$, we get the 8^{th} term of the given H.P. which is $\frac{1}{3\times 8-1} = \frac{1}{23}$.

Alternatively, a_8 of the $A.P.$ $= a_1 + (8-1)d$

$$= 2 + 7(3) = 23$$

Thus, the 8^{th} term of the given $H.P.$ $= \frac{1}{23}$

Example 24: If the 4^{th} term and 7^{th} term of the $H.P.$ are $\frac{2}{13}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution: Since the 4^{th} term of the $H.P.$ $= \frac{2}{13}$ and its 7^{th} term $= \frac{2}{25}$, therefore the 4^{th} and 7^{th} terms of the corresponding $A.P.$ are $\frac{13}{2}$ and $\frac{25}{2}$ respectively.

Now taking a_1, the first term and d, the common difference of the corresponding $A.P.$, we have,

$$a_1 + 3d = \frac{13}{2} \quad \text{(i)} \qquad \text{and} \qquad a_1 + 6d = \frac{25}{2} \quad \text{(ii)}$$

Subtracting (i) from (ii), gives

$$3d = \frac{25}{2} - \frac{13}{2} = 6 \quad \Rightarrow \quad d = 2$$

From (i), we get

$$a_1 = \frac{13}{2} - 3d = \frac{13}{2} - 6 = \frac{1}{2}$$

Thus, a_2 of the $A.P. = a_1 + d = \frac{1}{2} + 2 = \frac{5}{2}$

and a_3 of the $A.P. = a_1 + 2d = \frac{1}{2} + 2(2)$

$$= \frac{1}{2} + 4 = \frac{9}{2}$$

Hence the required $H.P.$ is $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$

6.9.1 Harmonic Mean (H.M.)

A number H is said to be the harmonic mean (H.M.) between two numbers a and b if a, H, b are in $H.P.$

Let a, b be the two numbers and H be their H.M. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

Therefore, $$\frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b}}{2} = \frac{\frac{b+a}{ab}}{2} = \frac{a+b}{2ab}.$$

and $$H = \frac{2ab}{a+b}$$

For example, $H.M.$ between 3 and 7 is

$$\frac{2 \times 3 \times 7}{3+7} = \frac{2 \times 21}{10} = \frac{21}{5}$$

6.9.2 *n* Harmonic Means between two Numbers

$H_1, H_2, H_3, \cdots, H_n$ are called n harmonic means (H.Ms.) between a and b if $a, H_1, H_2, H_3, \cdots, H_n, b$ are in H.P. If we want to insert n H.Ms., between a and b, we first find n $A.Ms$ $A_1, A_2, \ldots, A_n$ between $\frac{1}{a}$ and $\frac{1}{b}$, then take their reciprocals to get n H.Ms. between a and b, that is, $\frac{1}{A_1}, \frac{1}{A_2}, \ldots, \frac{1}{A_n}$ will be the required n H.Ms. between a and b.

Example 25: Find three harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

Solution: Let A_1, A_2, A_3 be three *A.Ms.* between 5 and 17, that is,

5, A_1, A_2, A_3, 17 are in *A.P.*

Using $a_n = a_1 + (n-1)d$, we get

$17 = 5 + (5-1)d$ $\quad (\because\ a_5 = 17$ and $a_1 = 5)$

$4d = 12$

$\Rightarrow$ $d = 3$

Thus, $A_1 = 5 + 3 = 8$, $A_2 = 5 + 2(3) = 11$ and $A_3 = 5 + 3(3) = 14$

Hence $\frac{1}{8}, \frac{1}{11}, \frac{1}{14}$ are the required harmonic means.

EXERCISE 6.9

1. Find the 9th term of the following harmonic sequences:
   (i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ (ii) $\frac{-1}{5}, \frac{-1}{3}, -1, \ldots$
2. Insert five harmonic means between the following given numbers:
   (i) $\frac{-2}{5}$ and $\frac{2}{13}$ (ii) $\frac{1}{4}$ and $\frac{1}{24}$
3. The first term of an *H.P.* is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.
4. If 5 is the harmonic mean between 2 and b, find b.
5. If the numbers $\frac{1}{k}, \frac{1}{2k+1}$ and $\frac{1}{4k-1}$ are in harmonic sequence, find k.
6. Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be *H.M.* between a and b.
7. If a^2, b^2 *and* c^2 are in *A.P.* show that $a + b$, $c + a$ and $b + c$ are in *H.P.*
8. If the H.M. and A.M. between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers.
9. If the (positive) G.M. and H.M. between two numbers are 4 and $\frac{16}{5}$, find the numbers.
10. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., show that a, b, c are in H.P.

11. If a, b, c, d are in H.P., show that $3(a-b)(c-d)=(b-c)(a-d)$.
12. If between any two numbers there are inserted two A.Ms. A_1, A_2, two G.Ms. G_1, G_2 and two H.Ms. H_1, H_2; show that $\frac{A_1+A_2}{G_1G_2}=\frac{H_1+H_2}{H_1H_2}$.
13. The H.M. of two numbers is 4. The A.M., A and G.M., G satisfy the relation $2A+G^2=27$. Find the numbers.
14. First three of the four numbers a, b, c, d are in A.P., and the next three are in H.P., show that $ad=bc$.
15. If a, b, c are in G.P., show that $\log_a x, \log_b x, \log_c x$ are in H.P.
16. If a, b, c are in H.P., show that

    (i) $\frac{a-b}{b-c}=\frac{a}{c}$ (ii) $(a-c)^2=(a+c)(a-2b+c)$.
17. If $2+x$, $5+x$ and $9+x$ are in H.P., find the value of x.
18. If the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are equal, prove that a, b, c are in H.P.

6.10 Miscellaneous Series

The Greek letter Σ(sigma) is used to denote sums of different types. For example, the notation $\sum_{i=m}^{n} a_i$ is used to express the sum $a_m+a_{m+1}+a_{m+2}+\ldots+a_n$ and the sum expression 1 + 3 + 5 + … to n terms is written as $\sum_{k=1}^{n}(2k-1)$, where $2k-1$ is the k^{th} term of the sum and k is called the index of summation. "1" and n are called the lower limit and upper limit of summation respectively.

The sum of the first n natural numbers, the sum of squares of the first n natural numbers and the sum of the cubes of the first n natural numbers are expressed in sigma notation as:

$$1+2+3+\ldots+n=\sum_{k=1}^{n} k \; ; \; 1^2+2^2+3^2+\ldots+n^2=\sum_{k=1}^{n} k^2 \; ; \; 1^3+2^3+3^3+\ldots+n^3=\sum_{k=1}^{n} k^3$$

We evaluate $\sum_{k=1}^{n}[k^m-(k-1)^m]$ for any positive integer m and shall use this result to find out formulae for three expressions stated above.

$$\sum_{k=1}^{n}[k^m-(k-1)^m]=(1^m-0^m)+(2^m-1^m)+(3^m-2^m)+\cdots$$
$$+[(n-1)^m-(n-2)^m]+[n^m-(n-1)^m]=n^m$$

i.e., $\sum_{k=1}^{n}[(k^m-(k-1)^m]=n^m$

If $m=1$, then $\sum_{k=1}^{n}[(k^1-(k-1)^1]=n^1$ i.e., $\sum_{k=1}^{n}1=n$

If $m=2$, then $\sum_{k=1}^{n}\left[k^2-\left(k-1\right)^2\right]=n^2$

Properties of Summation:

(i) $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$

(ii) $\sum_{k=1}^{n}\alpha a_k=\alpha\sum_{k=1}^{n}a_k$

To Find the Formulae for the Sums

(i) $\sum_{k=1}^{n}k$ (ii) $\sum_{k=1}^{n}k^2$ (iii) $\sum_{k=1}^{n}k^3$

(i) We know that $(k-1)^2=k^2-2k+1$ and this identity can be written as:

$$k^2-(k-1)^2=2k-1 \qquad \text{(A)}$$

Taking summation on both sides of (A) from $k=1$ to n, we have

$$\sum_{k=1}^{n}[(k^2-(k-1)^2] = \sum_{k=1}^{n}(2k-1)$$

i.e., $$n^2 = 2\sum_{k=1}^{n}k-n \qquad (\because \sum_{k=1}^{n}1=n)$$

or $$2\sum_{k=1}^{n}k = n^2+n$$

Thus $$\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$$

Similarly, we can prove easily

(ii) $\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$ (iii) $\sum_{k=1}^{n}k^3=\left[\frac{n(n+1)}{2}\right]^2$

Example 26: Find the sum of the series $1^3+3^3+5^3+\ldots$ to n terms.

Solution: $T_k=(2k-1)^3 \qquad (\because 1+2(k-1)=2k-1)$

$$=8k^3-12k^2+6k-1$$

Let S_n denote the sum of n terms of the given series, then

$$S_n=\sum_{k=1}^{n}T_k$$

or $$S_n=\sum_{k=1}^{n}(8k^3-12k^2+6k-1)$$

$$= 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= 8\left[\frac{n(n+1)}{2}\right]^2 - 12\left[\frac{n(n+1)(2n+1)}{6}\right] + 6\left[\frac{n(n+1)}{2}\right] - n$$

$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$

$$= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + n(3n+3) - n$$

$$= 2n[(n^3+2n^2+n) - (2n^2+3n+1)] + n(3n+3-1)$$

$$= 2n[(n^3-2n-1) + n(3n+2)$$

$$= 2n(n^3-2n-1) + n(3n+2)$$

$$= n[2n^3 - 4n - 2 + 3n + 2]$$

$$= n[2n^3 - n] = n[n(2n^2-1)]$$

$$= n^2[2n^2-1]$$

Example 27: Find the sum of n terms of series whose n^{th} terms is $n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + 1$.

Solution: Given that

$$T_n = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + 1$$

Thus $\quad T_k = k^3 + \frac{3}{2}k^2 + \frac{1}{2}k + 1$

and $\quad S_n = \sum_{k=1}^{n}\left(k^3 + \frac{3}{2}k^2 + \frac{1}{2}k + 1\right)$

$$= \sum_{k=1}^{n} k^3 + \frac{3}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{n^2(n+1)^2}{4} + \frac{3}{2}\times\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\times\left[\frac{n(n+1)}{2}\right] + n$$

$$= \frac{n}{4}[n(n^2+2n+1) + (2n^2+3n+1) + (n+1) + 4]$$

$$= \frac{n}{4}(n^3+2n^2+n+2n^2+3n+1+n+1+4)$$

$$= \frac{n}{4}(n^3+4n^2+5n+6)$$

1. Sum the following series upto n terms.

(i) $1 \times 1 + 2 \times 4 + 3 \times 7 + \dots$

(ii) $1 \times 3 + 3 \times 6 + 5 \times 9 + \dots$

(iii) $1 \times 4 + 2 \times 7 + 3 \times 10 + \dots$

(iv) $3 \times 5 + 5 \times 9 + 7 \times 13 + \dots$

(v) $1^2 + 3^2 + 5^2 + \dots$

(vi) $2 \times 1^2 + 4 \times 2^2 + 6 \times 3^2 + \dots$

(vii) $3 \times 2^2 + 5 \times 3^2 + 7 \times 4^2 + \dots$

(viii) $1 + (1 + 2) + (1 + 2 + 3) + \dots$

(ix) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

(x) $2 \times 4 \times 7 + 3 \times 6 \times 10 + 4 \times 8 \times 13 + \dots$

2. Sum the series.

(i) $1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2$

(ii) $\frac{1^2}{1} + \frac{1^2 + 2^2}{2} + \frac{1^2 + 2^2 + 3^2}{3} + \dots$ to n terms

3. Find the sum to n terms of the series whose n^{th} terms are given.

(i) $3n^2 + n + 1$

(ii) $n^2 + 4n + 1$

4. Given n^{th} terms of the series, find the sum to $2n$ terms.

(i) $3n^2 + 2n + 1$

(ii) $n^3 + 2n + 3$

6.11 Real Life Problems involving Sequences and Series

Example 28: Vehicle Arrival Sequence

Vehicles arrive at a toll booth at a rate of 4 cars every 5 minutes. Represent the number

Simple Interest on Loan (Arithmetic Sequence with Particular Term)

Example 29: To buy furniture for a new apartment Tayyab borrowed Rs. 50,000 at 8% simple interest for 11 years. How much interest will he pay?

Solution: Since 8% is the yearly interest rate, we have

$$\text{Interest after one year} = \text{Rs. } 50{,}000 \times \frac{8}{100} \times 1 = \text{Rs. } 4000$$

$$\text{Interest after two years} = \text{Rs. } 50{,}000 \times \frac{8}{100} \times 2 = \text{Rs. } 8000$$

Therefore, we have the A.P.

$$4000, 8000, 12000, \ldots$$

Here, $a_1 = 4000$, $a_2 = 8000$, $d = a_2 - a_1 = 4000$, $n = 11$

Using the formula

$$a_n = a_1 + (n-1)d$$

$$a_{11} = 4000 + (11-1)(4000)$$

$$= 4000 + 10(4000)$$

$$= 4000 + 40000$$

$$= \text{Rs. } 44000$$

Thus, Tayyab will pay a total interest of Rs. 44000 on borrowed amount of Rs 50,000 after 11 years.

Compound Interest on Loan (Geometric Sequence with Particular Term)

Example 30: Amna invests Rs. 200000 at 5% interest compounded annually. What total amount will she get after 10 years?

Solution: Let the principal amount be P. Then,

$$\text{The interest for the first year} = P \times \frac{5}{100} = P(0.05)$$

$$\text{The total amount after first year} = P + P(0.05) = P(1 + 0.05)$$

$$\text{The interest for the second year} = P(1 + 0.05) \times 0.05$$

$$\text{The total amount after second year} = P(1 + 0.05) + P(1 + 0.05) \times 0.05$$

$$= P(1 + 0.05)(1 + 0.05)$$

$$= P(1 + 0.05)^2$$

Similarly, the total amount after third year $= P(1 + 0.05)^3$

Thus, we have sequence of amounts

$$P(1.05), P(1.05)^2, P(1.05)^3, \ldots$$

which is clearly a G.P., with

$$a_1 = P(1.05),\ r = 1.05,\ n = 10,\ a_{10} = ?$$

Using the geometric sequence formula

$$a_n = a_1 r^{n-1}$$

$$a_{10} = a_1 r^{10-1}$$

$$= P(1.05) \times (1.05)^9$$

$$= (200000)(1.05)^{10} \qquad \because P = 200000$$

$$= (200000)(1.62889)$$

$$= 325778.92$$

Thus, the total amount Amna will get after 10 years will be Rs. 325778.92

Grid Column Distribution (Arithmetic Series Sum of Terms)

Example 31: A web designer is using a 12-column grid system where each column increases in width by 10px from the previous one. The first column width is 50px wide. Find the total width occupied by all 12 columns.

Solution: This follows an arithmetic series with:

First term = a_1 = 50, Common difference = 10

Number of terms = n = 12

Using the formula for the sum of an arithmetic series:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(50) + (12-1)(10)]$$

$$= 6[100 + 110] = 6\,[210]$$

$$= 1260px$$

Thus, the total width of all 12 columns is 1260px.

Example 32: Motor Vehicle Leasing Using Arithmetic Sequence

A company leases a motor vehicle with the following terms:

- The first monthly payment is Rs. 15,000
- Each subsequent payment increases by Rs. 500 due to inflation adjustments.
- The lease term is 24 months.

Find:

(i) What is the payment in the 24th month?

(ii) What is the total amount paid over 24 months?

(iii) If the company can only afford to pay a total of Rs. 400,000, can they complete the 24-months lease?

(iv) Find maximum months n such that total, payment $S_n \le 400,000$.

Solution: Given:

First term = a_1 = 15000

Common difference = d = 500

Number of terms = n = 24

(i) Payment in 24[th] month:

Using the formula

$$a_n = a_1 + (n-1)d$$
$$a_{24} = 15000 + (24-1)(500)$$
$$= 15000 + 23 \times 500$$
$$= 15000 + 11500 = \text{Rs. } 26500$$

(ii). Total payment over 24 months using the formula

$$S_n = \frac{n}{2}(a_n + a_n)$$
$$= \frac{24}{2}(15000 + 26500) = 12(41500) = \text{Rs. } 498000$$

(iii) Can the company afford the lease? No. Total payments (Rs. 498000) exceed the budget of Rs. 400,000 by Rs. 98,000.

(iv) Using: $S_n = \frac{n}{2}[2a_1 + (n-1)d] \leq 400,000$

Substituting the values:

$$\frac{n}{2}[2(15000) + (n-1)(500)] \leq 400,000$$
$$n\,[15000 + 250n - 250] \leq 400,000$$
$$n(250n + 14750) \leq 400,000$$
$$250n^2 + 14750n - 400000 \leq 0$$
$$n^2 + 59n - 1600 \leq 0$$

Associated equation is $n^2 + 59n - 1600 = 0$

$$n = \frac{-59 \pm \sqrt{(59)^2 - 4(1)(-1600)}}{2(1)}$$
$$n = \frac{-59 \pm 99.4}{2}$$
$$n = \frac{-59 - 99.4}{2},\ n = \frac{-59 + 99.4}{2}$$

$n = -79.2$, $n = 20.2$

Clearly $n = 20$ satisfy the inequality.

So, $n = 20$ is the maximum months such that payment $S_n \leq 400,000$.

EXERCISE 6.11

1. A sum of Rs.10400 is paid off in 40 instalment such that each instalment is Rs.10 more than the preceding instalment. Calculate the value of the first instalment.
2. An investor invests Rs. 150000 at an annual compound interest rate of 6% for 8 years. Find the total amount will he get after 8 years.
3. The population of a town is 4084101 at present and five years ago it was 3200000. Find its rate of increase if it increased geometrically.
4. Determine the total worth of a yearly Rs. 5000 investment after 20 years if the interest rate is 5% compounded annually.
5. A water tank develops a leak. Each week, the tank loses 5 gallons of water due to the leak. Initially, the tank is full and container 2000 gallons.
   (a) How many gallons are in the tank 20 weeks later?
   (b) How many weeks until the tank is half-full?
   (c) How many weeks until the tank is empty?
6. A drug company has manufactured 7 million doses of a vaccine to date. They promise additional production at a rate of 1.4 million doses/month over the next year.
   (a) How many doses of the vaccine, in total, will have been produced after a year?
   (b) The general term a_n describes the total number of doses of the vaccine produced. Describe the meaning of the variable n in the context of this problem. Find the general term a_n
   (c) Find the value of a_{10} and interpret its meaning in words.
7. At a toll booth, the number of vehicles passing through during the first minute is 100. Due to road congestion, each minute only 80% of the vehicles from the previous minute manage to pass.
   (a) Represent the number of vehicles passing each minute as a sequence.
   (b) Find the total number of vehicles that pass through in 15 minutes.
   (c) What is the maximum number of vehicles that can pass in the long run (as time $t \to \infty$)
8. A sum of Rs. 5000 is inverted at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an A.P.? If so find the interest at the end of 20 years making use of this fact.

9. A machine is purchased for Rs.20,000. Depreciates at 6% per annum for the first four years and after that 8% per annum for the next six years. Depreciation being calculated on diminishing value. Find the value of the machine after a period of 10 years.

10. Two cars start together in the same direction from the same place. The first goes with uniform speed of 20km/h. The second goes at a speed of 12km/h in the first hour and increase the speed by 1 km/h each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?

11. 150 workers were engaged to finish a piece of work in a certain number of days. Five workers dropped the second day, five more workers dropped the third day and so on. It takes 10 more days to finish the work now. Find the number of days in which the work was completed.

12. A radioactive product has a half life of 5 years. If the radioactivity level is 68 microcuries after 20 years. Determine the original level of radioactivity.

13. An object moving in a line is given an initial velocity of 4.5 m/s and a constant acceleration of 2.5 m/s^2. How long will it take the object to reach a velocity of 20m/s?

14. In an integrated circuit with an initial current of 1080 mA, the temperature in the

Unit 7 Permutation and Combination

INTRODUCTION

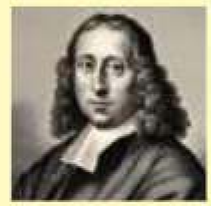
In our daily life, permutation and combination play vital role in counting total number of possibilities, in arrangements and selections of objects or things. Permutation and combination are used in many fields of sciences. For example,

- In probability theory, permutation and combination are used to compute how many times an event occurs in various scenarios and used to estimate the odds of winning a lottery.
- In biology, these are used to find out the total numbers of possible DNA sequences.
- In computer science, these are used to count the possible number of passwords of a given length by using some specific characteristics.
- Moreover, these are the important parts of many encryption algorithms to ensure the privacy and integrity of a data set.

History

Augustin Louis Cauchy (1789 – 1857) is the father of permutation.

Blaise Pascal and Pierre de Fermat (1607-1665) gave an idea to generate the combinations of objects.



Pascal and Leibniz are the founder of modern combinatorics.

7.1 Fundamental Principle of Counting

Danish wants to prepare invitation cards of 5 different colours (red, blue, green, orange and yellow) by changing any of 3 shapes (circle, square and rectangle). How many cards can Danish make?

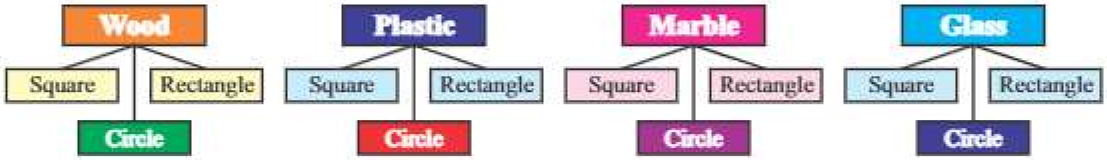
Challenge!
Make a tree diagram and find how many cards can Danish make?

The problem is to count the total number of ways in which Danish can make cards. One way to find the solution is by making tree diagram.

Let us discuss another scenario: Danish's father wants to buy a table and has asked his son to help him decide. He narrowed down his options for manufacturer, types of material (wood, plastic, glass and marble) and types of shape (circle, square and rectangle). Find the total number of table choices from the above options.

Again the problem is to count the total number of ways in which Danish's father can choose a table.

1st Way: By making tree diagram.



From tree diagram, it is clearer there are 12 choices for Danish's father to buy a table with one type of material and one type of shape.

2nd Way: By multiplying, Danish's father can find the total number of table choices to buy a table with one kind of material and shape.

Total number of table choices = Total types of material × Total types of shape
= 4 × 3 = 12 choices

These examples show that when making a choice involving multiple stages or categories, we can find the total number of outcomes by multiplying the number of options at each stage.

Statement

Suppose A and B are two events, the event "A" occurs in "m" different ways, and the event "B" occurs in "n" different ways then the total number of ways that the two events together can occur is the product of "m" and "n".

$$\text{Total number of ways} = mn$$

Proof: Let $A = \{a_1, a_2, a_3, \cdots, a_m\}$ and $B = \{b_1, b_2, b_3, \cdots, b_n\}$. Let P denotes the event that both events A and B occur together then $P[(a_i, b_j): a_i \in A,\ b_j \in B, 1 \leq i \leq m, 1 \leq j \leq n\} = A \times B$. Hence the number of ways in which both events A and B can occur is the number of elements in $A \times B$ which is mn.

This principle can be extended to three or more events. For instance, if event A can occur in m ways, event B can occur in n ways and event C can occur in k ways, the number of ways that three events can occur all together is the product of m, n and k.

Try yourself

If three dice are rolled together, how many total numbers of ways occur?

$$\text{Total number of ways} = m \times n \times k$$

Factorial (!)

Suppose there are four chairs to be occupied by four students and we are interested in counting all the possible ways the students can be seated.

To occupy the first chair there are 4 options. For the second chair, only 3 students remain, so there are 3 options. Similarly, for the third and fourth chairs, there are 2 and 1 options respectively.

History

The factorial notation (!) was introduced by Christian Kramp (1760-1826) in 1808

This notation is frequently used to solve permutation and combination.

In this way, we have to perform four independent events with 4, 3, 2, and 1 options respectively.

By the **Fundamental Principle of Counting**, the total number of ways to occupy all the chairs is 4.3.2.1 = 24

Such problems frequently occur in daily life, where we to multiply the first n natural numbers: $1, 2, 3, \cdots, n$.

We call this product the factorial of n and denote it by $n!$ Or $\lfloor n$, thus for a natural number n:

$n! = \lfloor n = n(n-1)(n-2)\cdots 3.2.1$

For some reason we also define $0! = 1$. In general if n is a non-negative integer, then its factorial is denoted and defined as

$$n! = \lfloor n = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)(n-2)\ \ldots\ 3.2.1 & \text{if } n \geq 1 \end{cases}$$

For example,
$1! = 1$
$2! = 2.1 = 2$
$3! = 3.2.1 = 6$
$4! = 4.3.2.1 = 24$
$5! = 5.4.3.2.1 = 120$
$6! = 6.5.4.3.2.1 = 720$

Challenge!

Can you find out $\frac{8!}{3!}$?

It can be easily observed that

$n! = n(n-1)!$ for $n \geq 1$

Example 1: Evaluate $\frac{8!}{6!}$

Solution: $\frac{8!}{6!} = \frac{8.7.6.5.4.3.2.1}{6.5.4.3.2.1} = 56$

Example 2: write 8.7.6.5 in the factorial form.

Solution: $8.7.6.5 = \frac{8.7.6.5.4.3.2.1}{4.3.2.1} = \frac{8!}{4!}$

Example 3: Evaluate $\frac{9!}{6!\,3!}$

Solution: $\frac{9!}{6!\,3!} = \frac{(9.8.7)6!}{6!(3.2.1)} = 84$

or $\frac{9!}{6!\,3!} = \frac{9.8.7.6.5.4.3!}{6.5.4.3.2.1.3!} = 84$

or $\frac{9!}{6!\,3!} = \frac{9.8.7.6.5.4.3.2.1}{6.5.4.3.2.1.3.2.1} = 84$

EXERCISE 7.1

1. Let us make paratha roll. We can choose our fillings from the following:
   Meat: Chicken or beef **Vegetable:** Onions, tomatoes or cucumber
   Sauce: Mayo or Chutney
   How many different kinds of rolls can we make?

2. Suppose we have 3 universities, and each offers 4 careers. Use a tree diagram to figure out how many possible career paths you can take.

3. Evaluate each of the following:

(i) $7!$ (ii) $9!$ (iii) $\frac{10!}{8!}$ (iv) $\frac{12!}{9!}$

(v) $\frac{9!}{2!\,7!}$ (vi) $\frac{5!}{2!\,3!\,1!}$ (vii) $\frac{10!}{3!4!}$ (viii) $\frac{12!}{3!3!5!}$

(ix) $\frac{12!}{3!(12-3)!}$ (x) $\frac{20!}{20!(20-20)!}$ (xi) $\frac{8!}{0!}$ (xii) $6!.0!.2!$

4. Write each of the following in the factorial form:

(i) $8.7.6.5$ (ii) $15.14.13.12.11$ (iii) $19.18.17.16$

(iv) $\frac{11.10.9.8.7}{5.4}$ (v) $\frac{10.9.8.7.6}{5.4.3.2.1}$ (vi) $\frac{50.49.48.47}{5.4.3.2.1}$

(vii) $n(n-1)(n-2)(n-3)$ (viii) $(n+2)(n+1)(n)(n-1)$

(ix) $\frac{(n+3)(n+2)(n+1)(n)}{5.4.3.2.1}$ (x) $n(n-1)(n-2)\ldots(n-r+2)$

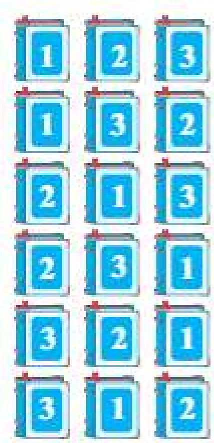
7.2 Permutation

One important application of the fundamental principle of counting is to determine the number of ways that objects can be arranged in order.

Definition: An arrangement of all or part of set of objects in a specific order is called a permutation. Number of permutations of $r(\leq n)$ objects taken from a set of n objects is written as ${}^nP_r$ or $P(n, r)$.

$${}^nP_r = \frac{n!}{(n-r)!} \quad \text{when } r \leq n$$

According to fundamental principle of counting:



(i) Three books of mathematics for grades 1, 2 and 3 can be arranged in a row taken all at a time (If books are distinct)

$${}^nP_r = {}^3P_3 \qquad \because\ n = r$$

$$= \frac{3!}{(3-3)!} = \frac{3!}{0!} \qquad \because\ 0! = 1$$

$$= 3! = 3\cdot 2\cdot 1 = 6 \text{ ways}$$

(ii) Number of ways of writing the letters of the WORD taken all at a time

$^nP_r = {}^4P_4 \quad \because \quad n = r$

$= \dfrac{4!}{(4-4)!} = \dfrac{4!}{0!} \quad \because \quad 0! = 1$

$= 4! = 4.3.2.1 = 24$ ways

n = Total number of things/objects
r = The number of selected things / objects

Challenge!

Can you make total number of permutations for the "WORD" pictorially?

Do you know!

In 1974, "Erno Rubik" invented a popular puzzle, each turn of the puzzle shows a permutation of the different colours. The name of this puzzle is "Rubik's Cube".

Theorem: Prove that: $^nP_r = n(n-1)(n-2)....(n-r+1) = \dfrac{n!}{(n-r)!}$

Proof: As there are n different objects to fill up r places. So, the first place can be filled in n ways. Since repetitions are not allowed, so after placing one object we are left with $(n-1)$ objects, thus the second place can be filled in $(n-1)$ ways. Similarly the third place can be filled in $(n-2)$ ways, and so on. This continues until the r^{th} place which can be filled in $n-(r-1) = n-r+1$ ways. Therefore, by the **Fundamental Principle of Counting**, r places can be filled by n different objects in $n(n-1)(n-2)....(n-r+1)$ ways.

$$^nP_r = n(n-1)(n-2)....(n-r+1)$$

$$= \frac{n(n-1)(n-2)...(n-r+1)(n-r)!}{(n-r)!}$$

$$^nP_r = \frac{n!}{(n-r)!}$$

Example 4: How many different 4-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, when no digit is repeated?

Solution: The total number of digits = 6

The digits forming each number = 4.

So, the required number of 4-digit numbers is given by:

$$^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 6.5.4.3 = 360$$

Example 5: In how many ways can a set of 4 different mathematics books, 3 different physics books and 2 different chemistry books be placed on a shelf with a space for 9 books, if:

(a) All the books are kept without any restriction.

(b) All the books of the same subject are kept together.

(c) Only the mathematics books are kept together.

Solution:

(a) All the books are kept without any restriction.

Total number of books $= 4 + 3 + 2 = 9$

$^{9}P_{9} = 9! = 9.8.7.6.5.4.3.2.1$

$= 362880$ ways

> **Reason for defining 0! = 1**
>
> If $n = r$, then $^{n}P_{n} = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$
>
> $n(n-1)\ \dots\ 3.2.1 = n! = \dfrac{n!}{0!}$ $\Rightarrow\ 0! = 1$

(b) All the books of the same subject are kept together.

$^{4}P_{4}\ .\ ^{3}P_{3}\ .\ ^{2}P_{2}\ .\ ^{3}P_{3} = 4!\ .\ 3!\ .\ 2!\ .\ 3!$

$= 3!\ .\ 24\ .\ 6\ .\ 2\ .\ 6$

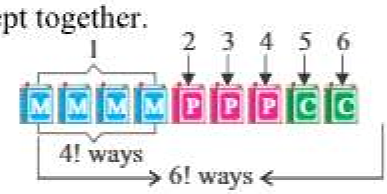
$= 1728$ ways

(c) Only the mathematics books are kept together.

$^{4}P_{4}\ .\ ^{6}P_{6} = 4!\ .\ 6!$

$= 24\ .\ 720$

$= 17280$ ways



Example 6: In how many ways 5 people are to be seated on a bench if:

(a) there are no restrictions

(b) two people can sit next to each other

(c) two people cannot sit next to each other.

> **Challenge!**
>
> Find the number of ways if only physics books are kept together.

Solution:

(a) when there is no restriction, then

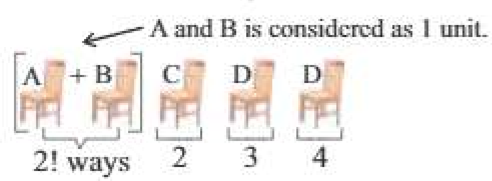
Number of ways $= {}^{5}P_{5} = 5! = 120$

(b) when two people can sit next to each other, then

$= {}^{4}P_{4}\ .\ ^{2}P_{2}$

$= 4! \cdot 2! = 24 \cdot 2$

$= 48$ ways



(c) when two people cannot sit next to each other, then

$= {}^{5}P_{5} -$ [2 can sit next to each other]

$= 5! - 48 = 120 - 48$

$= 72$ way

> **Try yourself**
>
> In how many ways 6 people are to seated on a table if 3 cannot sit next to each other?

EXERCISE 7.2

1. Evaluate the following:

   (i) ${}^{10}P_5$ (ii) ${}^{5}P_2$ (iii) ${}^{7}P_7$ (iv) ${}^{10}P_3$

2. Find the value of n when:

   (i) ${}^{n}P_3 = 504$, (ii) ${}^{15}P_n = 15.14.13.12\ 11$ (iii) ${}^{n}P_5 : {}^{n-2}P_2 = 540 : 1$

3. Prove from the first principle that:

   (i) ${}^{n}P_r = n\ .\ {}^{n-1}P_{r-1}$ (ii) ${}^{n}P_r = {}^{n-1}P_r + r\ .\ {}^{n-1}P_{r-1}$

4. How many signals can be given by 6 flags of different colours, using 2 flags at a time?

5. From a deck of 13 cards, find in how many ways these are arranging in a rectangular form? Hint (order is matter)

   (i) All cards (ii) 8 cards (iii) 10 cards

6. In how many ways can the seven alphabet a e f i o u h be arranged in a row?

7. There are 8 men. Find the number of ways of arranging them in a row if:

   (a) Two old men are at left side

   (b) The youngest man is not at the right side

8. How many arrangements are there, if 6 books are arranged in a row out of 12 books?

9. Find permutation of 10 people sitting on a bench if:

   (a) There are no restriction (b) 3 cannot sit next to each other.

10. In how many ways can a set of 4 different blue pens, 3 different red pens and 6 black pens be placed in a rectangular form rack with a space for 10 pens if:

    (a) All the pens are placed without any restriction

    (b) All the pens of the same colour are placed together

    (c) Only the red pens are placed together

11. Hamza wants to distribute 15 pencils among 6 needy children in this way that the youngest gets 4 pencils and others get 2 pencils. Find how many ways, there are of arranging in a row form?

12. In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together?

13. Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subject are together.

14. In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?

7.3 Permutation of Objects Not All Different

Suppose we have to find the permutations of the letters of the word BITTER using all the letters in it. The word BIT_1T_2ER consists of 6 different letters which can be permuted among themselves in 6! ways.

We can see that all the letters of the word BITTER are not different. It has 2Ts in it. After replacing 2Ts, we can see there are 2! ways.

The replacement of the two Ts by T_1 and T_2 in any other permutation will give rise to 2 permutations.

Hence, the number of permutations of the letters of the word BITTER taken all at a time.

$$\frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 360 \text{ ways}$$

Remember!

If there is n_1 alike objects of one kind, n_2 alike objects of second kind and n_3 alike objects of third kind, then the number of permutations of n objects taken all at a time is given by:

$$\frac{n!}{n_1!\,.\,n_2!\,.\,n_3!} = \binom{n}{n_1,\, n_2,\, n_3}$$

Example 7: In how many ways can be letters of the word MISSISSIPPI be arranged when all the letters are to be used?

Solution: Total number of letters in the word =11

MISSISSIPPI

I is repeated 4 times = 4! ways

S is repeated 4 times = 4! ways

P is repeated 2 times = 2! ways

M comes once only = 1! ways

$$\text{Required number of permutations} = \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!} = 34650 \text{ ways}$$

Circular Permutation

The permutation in which the objects are arranged in a circular order is known as circular permutation.

Note:

The following circular arrangements are reflection of each other and considered same when anticlockwise and clockwise arrangements are considered identical.

Circular permutation can occur in two cases:

Case-I: When clockwise and anticlockwise arrangements are considered different

In a linear arrangement, changing the order of objects results in a new arrangement. However, in a circular arrangement, rotating the entire circle does not produce a new, distinct arrangement.

For example, suppose three people A, B, and C are sitting around a round table. The following three linear arrangements

A – B – C, B – C – A and C – A – B are all considered the same in circular permutations because each one is simply a rotation of the others.

We conclude that:

3 linear permutations gives 1 circular permutation.

3! linear permutations gives $\frac{1}{3}.3! = \frac{3!}{3} = 2!$ permutations.

Generalizing the above idea if n objects are arranged in a circle, the number of distinct circular permutations is $\frac{n!}{n} = (n-1)!$

Case-II: When clockwise and anticlockwise arrangements are considered identical

In many real-life situations, a circular permutation and its mirror image are not considered different.

For example, if three beads red, blue, and black are arranged in a necklace, then an arrangement and its reflection (as shown in the figure) are considered the same.

In such cases, we divide the total number of circular permutations by 2 to eliminate symmetrical duplicates.

Thus, the number of distinct circular permutations is:

$$\frac{(n-1)!}{2}$$

Example 8: In how many ways can 4 persons be seated at a round table, while:

(i) clockwise and anticlockwise orders are different

(ii) clockwise and anticlockwise orders are identical.

Solution: Let A, B, C and D be the 4 persons.

(i) If clockwise and anticlockwise orders are different

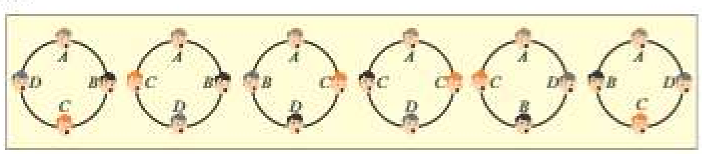
According to Case-I

The possible number of ways are:

$= (n-1)!$ ways

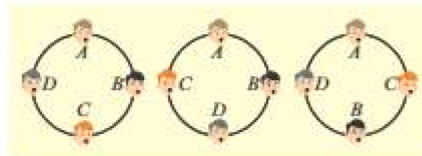
$= (4-1)! = 3!$

$= 3.2.1 = 6$ ways.



(ii) If clockwise and anticlockwise orders are identical

According to Case-II



The possible number of ways are $= \frac{(n-1)}{2}$ ways

$$= \frac{(4-1)!}{2} = \frac{3!}{2}$$

$$= \frac{3 \cdot 2}{2} = 3 \text{ ways}$$

EXERCISE 7.3

1. How many arrangements of the letters of the following words, taken all together can be made?

   (i) CURRICULUM (ii) ADSORPTIVELY (iii) PROBABILITY

2. A girl has 9 marbles. There are 4 red marbles, 3 blue, and 2 green marbles. If she arranges them in a row, then find in how many different arrangements she can make take all at time?

3. In how many different ways can the following persons sit in a round table?

   Hint (Solve according to both the cases)

   (a) 8 persons (b) 7 persons (c) 6 persons

4. In how many ways can 5 couples sit on a round table if no two women are sitting together?

5. How many arrangements of the letters of the word ATTACKED can be made if each arrangement begins with C and ends with K?

6. How many 6-digit numbers can be formed from the digits 7, 7, 8, 8, 9, 9?

7. 15 members of a club form 4 committees of 3, 5, 4, 3 members so that no member is a member of more than one committee. Find the number of committees.

8. The D.C.Os of 11 districts meet to discuss the law-and-order situation in their districts. In how many ways can they be seated at a round table, when two particular D.C.Os insist on sitting together?

9. The Governor of the Punjab calls a meeting of 14 officers. In how many ways can they be seated at a round table?

10. Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables. Guests of one sex sit at one round table and the guests of the other sex sit at the second table. Find the number of ways in which all guests are seated.

11. Find the number of ways in which 5 men and 5 women can be seated at a round table in such a way that no two persons of the same sex sit together.
12. In how many ways can 6 keys be arranged in a circular key ring?
13. How many necklaces can be made from 6 beads of different colours?

7.4 Combination

Suppose, a teacher uses the names of few students to make a team for a writing competition. Such as Ahmad, Sana, Hamza and Danish. As a combination of team members, (Ahmad, Sana, Hamza and Danish) is equivalent to (Hamza, Ahmad, Danish and Sana). Because same students are in the combination. Consequently, you have the same team because the order of the name of the students does not matter.

Ahmad	Sana	Hamza	Danish
Hamza	Ahmad	Danish	Sana

So, we are interested in the membership of the team and not in the ways the students are listed (arranged).

Definition

A combination of r objects taken out of n objects is a subset of r objects of a set of n objects.

The number of combinations of n different objects taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$ and is given by ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Theorem. Prove that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Proof: Elements of a subset of r objects of a set of n objects can be arranged among themselves in $r!$ ways. So, each combination will give rise to $r!$ permutation. Thus, there will be ${}^nC_r \times r!$ permutations of n different objects taken r at a time that is:

$${}^nC_r \times r! = {}^nP_r$$

$$\Rightarrow \quad {}^nC_r \times r! = \frac{n!}{(n-r)!} \qquad \therefore \quad {}^nC_r = \frac{n!}{r!(n-r)!}$$

Which completes the proof.

Need to know

Need to know!

The formulae ${}^nP_r$ and ${}^nC_r$ are also known as counting formulae. Because, they are used to count the possible number of ways without listing them all.

Corollary:

(i) If $r = n$, then ${}^nC_n = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!\,0!} = 1$

(ii) If $r = 0$, then ${}^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!\,n!} = 1$

7.4.1 Applications of Combination in Real Life

Example 9: Zain has 8 different fruits. He wants to select 5 fruits out of 8 fruits to make a fruit chart. How many combinations of fruits he can select?

Solution: To solve this problem, we have to find the number of combinations of 5 fruits out of 8 fruits. In this situation, $n = 8$ and $r = 5$.

$$^nC_r = \frac{n!}{r!(n-r)!}$$

After putting values

$$^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\,.\,3!}$$

$$= \frac{8\times 7\times 6\times 5!}{5!\,.\,3!} = \frac{8\times 7\times \cancel{6}}{\cancel{3\,.\,2}\,.\,1}$$

$$= 8\times 7 = 56 \text{ ways}$$

Zain has 56 different ways to select 5 different fruits to make a fruit chart.

Example 10: In a school, a class consists of 12 girls and 8 boys. The teacher wants to select 5 students for an activity. In how many ways can the students be selected including? (i) 2 girls (ii) 5 boys (iii) 2 boys

Solution: Number of girls = 12

Number of boys = 8

(i) Now let's find the total number of ways to select students when exactly 2 are girls.

$$\left({}^{12}C_2\right)\left({}^{8}C_3\right) = \frac{12!}{2!\,10!}\cdot\frac{8!}{3!\,5!} = \frac{12\cdot 11\cdot 10!}{2\cdot 10!}\cdot\frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 1\cdot 5!} = 3696$$

Challenge!

A restaurant offers 6 flavours of pizza. How many ways are there to select 2 flavours of pizza?

(ii) Let's find total number of ways to select students when exactly 5 students are boys.

$$^8C_5 = \frac{8!}{5!\,(8-5)!} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6\cdot 5!}{5!\,3\cdot 2\cdot 1} = 56$$

(iii) Let's find total number of ways to select students when exactly 2 students are boys.

$$\left({}^{8}C_2\right)\left({}^{12}C_3\right) = \frac{8!}{2!\,6!}\cdot\frac{12!}{3!\,9!} = \frac{8\cdot 7\cdot 6!}{2\cdot 6!}\cdot\frac{12\cdot 11\cdot 10\cdot 9!}{3\cdot 2\cdot 1\cdot 9!} = 36960$$

7.4.2 Complementary Combinations

Theorem. Prove that: $^nC_r = {}^nC_{n-r}$

Proof: If from n different objects, we select r objects then $(n - r)$ objects are left. Corresponding to every combination of r objects, there is a combination of $(n - r)$

objects and vice versa. Thus, the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

$$\therefore \quad {}^nC_r = {}^nC_{n-r}$$

$${}^nC_{n-r} = \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

$${}^nC_{n-r} = {}^nC_r$$

Note:

This result will be found useful in evaluating ${}^nC_r$ when $r > \frac{n}{2}$.

For example,

$${}^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}C_2 = \frac{(12).(11)}{2} = 6.11 = 66$$

Example 11: Find the number of the diagonals of a 6-sided figure.

Solution: A 6-sided figure has 6 vertices by joining any two vertices, we get a line segment.

$$\therefore \quad \text{Number of line segments} = {}^6C_2 = \frac{6!}{2!\,4!} = 15$$

But these line segments include 6 sides of the figure

$$\therefore \quad \text{number of diagonals} = 15 - 6 = 9$$

Difference between permutation and combination

Permutation	Combination
• Order is important. e.g., *ab* and *ba* are different (because order of any object is matter) • Arrangement of objects e.g. arrangement of: * ball of different colours * English alphabet (letters) * people while sitting on chairs	• Order is not important e.g., *ab* and *ba* are same (because order does not matter) • Selection of objects e.g. selection of: * different colours * members in a team * food items

Application of Permutations and Combinations in Cryptography

Example 12: Zain wants to generate a password for his laptop to secure the data. He can take only 6 characters to generate a password. Each character can either be an upper case letter $(A-Z)$ or digits from $(0-9)$.

Can you tell how many passwords can be generated by using the above letters and digits:

(i) If repetition of characters is not allowed

(ii) If repetition of characters is allowed

Solution: Total number of letters = 26

Total number of digits = 10

Total number of letters and digits = 26 + 10 = 36

n = total number of characters = 36

r = required number of characters = 6

(i) If repetition of characters is not allowed, we find out total possible permutations as.

$$^{n}P_{r} = {}^{36}P_{6} = \frac{36!}{(36-6)!} = \frac{36!}{30!}$$

$$= \frac{36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31\cdot 30!}{30!}$$

$$= 36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31$$

$$= 1,402,410,240 \quad \text{ways}$$

Hence, $1,402,410,240$ passwords can be generated by using the 26 alphabet and 10 digits. (If repetition of the characters is not allowed)

(ii) If the repetition of the characters is allowed. Using Fundamental Principle of Counting:

The total number of possible combinations = 36 × 36 × 36 × 36 × 36 × 36 = 36^6

Hence, 36^6 passwords can be generated by using the 26 alphabets and 10 digits, If repetition of characters is not allowed.

Application of permutations to estimate the odd of winning the lottery.

Example 13: A box contains 15 cards from (1 – 15). Danish is to select 5 cards. If all the selected cards are the first five multiples of 2 then Danish will win the game. Find Danish's chance of winning the game, when

(i) Order is important (ii) Order is not important

Solution: n = total number of cards = 15

r = required number of cards = 5

(i) When order is important,

$$\text{Total possible ways} = {}^{n}P_{r} = {}^{15}P_{5} = \frac{15!}{(15-5)!}$$

$$= \frac{15!}{10!} = 360,\ 360 \text{ ways}$$

Hence, Danish's chance to win the game = $\frac{1}{360,\ 360}$ = 0.000002775

(ii) When order is not important

n = Total number of cards = 15

r = Required number of cards = 5

$$\text{Total possible ways} = {}^nC_r = {}^{15}C_5 = \frac{15!}{5!(15-5)!}$$

$$= \frac{15!}{5!\,10!} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times \cancel{10!}}{5!\,.\,\cancel{10!}}$$

$$= \frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1} = 3003 \text{ ways}$$

Hence, Danish's chance to win the game = $\frac{1}{3003} = 0.00033$

Application of Permutation and Combination to choose different sets of songs for Certain Occasions

Example 14: On Independence Day, a DJ has a list of ten different national songs. He wants to select any five national songs for the day. Find how many ways he can select and play the songs:

(i) If the order of playing the songs matters

(ii) If the order of playing the songs does not matter.

Solution: (i) When order matters

n = total number of national songs = 10

r = required number of national songs = 5

$$\text{Total number of ways} = {}^nP_r = {}^{10}P_5$$

$$= \frac{10!}{(10-5)!} = \frac{10!}{5!} = 30,240 \text{ ways}$$

Hence, the DJ can play the five national songs in 30,240 different ways.

(ii) When order is not matter

n = total number of national songs = 10

r = total number of selected national songs = 5

$$\text{Total number of ways} = {}^nC_r = {}^{10}C_5 = \frac{10!}{5!(10-5)!}$$

$$= \frac{10!}{5!\,.\,5!} = 252 \text{ ways}$$

Hence, the DJ can play the five national songs in 252 different ways.

EXERCISE 7.4

1. Evaluate the following:

   (i) ${}^5C_3$ (ii) ${}^8C_5$ (iii) ${}^3C_3$ (iv) ${}^{10}C_7$

2. Find the value of n, when

   (i) ${}^nC_6 = {}^nC_2$ (ii) ${}^nC_{11} = \frac{14.13.12}{3!}$ (iii) ${}^nC_5 = {}^nC_{10}$

3. In how many ways can five subjects be selected out of eight subjects to select a course programme?

4. Find how many ways there are to choose vowel words from the letter of English alphabet?

5. In how many ways 3 dishes of Desi foods and 2 dishes of Chinese foods be selected from 6 dishes of desi foods and 8 dishes of Chinese foods?

6. From a standard deck of 52 playing cards, there are 26 black cards and 26 red cards. How many different possible ways are made of eight cards if select 3 cards of black colour and others are of red colour?

7. A bag contains 8 red balls, 7 green balls. Find the total number of possible ways in which five balls are selected in a way:

   (i) 3 red and 2 green (ii) 1 red and 4 green

   (iii) 4 red and 1 green (iv) All the red balls

8. How many (a) diagonals and (b) triangles can be formed by joining the vertices of the polygon having:

   (i) 5 sides (ii) 8 sides (iii) 12 sides?

9. The members of a club are 10 boys and 8 girls. In how many ways can a committee of 6 boys and 3 girls be formed?

10. How many committees of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

11. In how many ways can a hockey team of 11 players be selected out of 15 players? how many of them will include a particular player?

12. Show that: ${}^{20}C_7 + {}^{20}C_6 = {}^{21}C_7$

13. There are 6 men and 8 women members of a club. how many committees of seven can be formed?

    (i) 3 women (ii) at the most 3 women (iii) at least 5 women?

14. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

15. A locker of a bank is locked with four letters (A-Z). How many different passwords can be generated if:
    (a) repetition of the alphabets is allowed
    (b) repetition of the alphabets is not allowed

16. Using a cryptographic system, a password is generated with 8 characters. Each character can either be a lowercase letter (a–f) or a digit (0–5). How many passwords can be generated if each password must contain exactly 5 lowercase letters and 3 digits?
    (a) With repetition allowed (b) Without repetition.

17. An urn contains the first 15 English letters (A–O). Sania is to randomly select 3 letters from the urn. She will win the game if the selected letters are the first three vowel letters. Find the probability of Sania winning the game if:
    (a) The order of the vowel letters matters
    (b) The order of the vowel letters does not matter

18. On Defense Day, Teacher I prepares a list of 10 national songs, and Teacher II also prepares a separate list of 10 different national songs. The principal wants to select 3 songs from Teacher I's list and 3 songs from Teacher II's list. In how many ways can the songs be selected if:
    (i) The sequence of the selected songs matters
    (ii) The sequence of the selected songs does not matter.

Answers

EXERCISE 1.1

1. (i) i (ii) i (iii) i (iv) $-i$ **.4** (i) $\left(\frac{-4}{65},\frac{-7}{65}\right)$ (ii) $\left(\frac{\sqrt{2}}{7},\frac{\sqrt{5}}{7}\right)$ (iii) $(1,0)$

5. (i) $\frac{-27}{41}-\frac{38}{41}i$ (ii) $\frac{-17}{2}-\frac{7}{2}i$ (iii) $\frac{1}{2}+\frac{i}{2}$ (iv) $\frac{-44}{25}+\frac{117}{25}i$ **6.** $\frac{11}{13}-\frac{23}{13}i$

7. (i) $2\sqrt{145}$ (ii) $\sqrt{149}$ (iii) $\sqrt{1354}$ (iv) $109\sqrt{109}$

EXERCISE 1.2

1. (i) $x=-19, y=22$ (ii) $x=9, y=6$ (iii) $x=-11, y=28$ **2.** $x=14, y=9$

3. (i) $x=9, y=5$ or $x=-9, y=-5$ (ii) $x=12, y=2$ or $x=-12, y=-2$ (iii) $x=\frac{71}{500}, y=\frac{47}{500}$

4. $\alpha=-2$ **5.** $x=8, y=3,\ a=2, b=1$ **7.** (i) $3-4i$ or $-3+4i$ (ii) $3-i$ or $-3+i$

(iii) $3-6i$ or $-3+6i$ (iv) $12+5i$ or $-12-5i$ **8.** $\pm\left(5-2\sqrt{3}i\right)$ **9.** $x=\frac{1}{25}, y=\frac{-57}{25}$

10. $x=\frac{-24}{29}, y=\frac{31}{29}$ **11.** $\alpha=\frac{5}{2}$

EXERCISE 1.3

1. (i) $(a+i2b)(a-i2b)$ (ii) $(3a+i4b)(3a-i4b)$ (iii) $3(x+iy)(x-iy)$ (iv) $9(4x+i5y)(4x-i5y)$
(v) $(z-i)(z-i)$ (vi) $(z+3-2i)(z+3+2i)$ (vii) $(z+2-i)(z+2+i)$
(viii) $\left(z-\frac{11-3i}{2}\right)\left(z-\frac{11+3i}{2}\right)$

2. (i) $(z+2)\left(z-\left(1-i\sqrt{3}\right)\right)\left(z-\left(1+i\sqrt{3}\right)\right)$ (ii) $(z+3)\left(z-\frac{3-i3\sqrt{3}}{2}\right)\left(z-\frac{3+i3\sqrt{3}}{2}\right)$
(iii) $(z-2)(z-4i)(z+4i)$ (iv) $(z-2)(z+2)(z-2i)(z+2i)$
(v) $(z-2i)(z+2i)(z-1)(z+1)$ (vi) $\left(z+\sqrt{2}i\right)\left(z-\sqrt{2}i\right)\left(z+\sqrt{3}i\right)\left(z-\sqrt{3}i\right)$

3. Roots: $3, -3, 4i, -4i$ Linear factor: $(z+3)(z-3)(z+4i)(z-4i)$ **4.** (i) $z=\frac{3\pm i\sqrt{23}}{4}$

(ii) $z=2\pm i\sqrt{21}$ (iii) $z=3\pm i$ (iv) $z=-2\pm 3i$ (v) $z=-\frac{3}{2}\pm\frac{3}{2}i$ (vi) $z=\frac{5\pm i\sqrt{41}}{6}$

5. $x=-2z^3+6z^2-8z+24$ **6.** $x=10z^4+30z^2-40$ **7.** $x=-3z^4+6z^3+42z^2-96z+96$

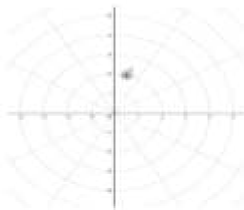
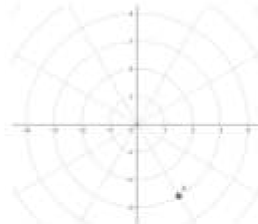
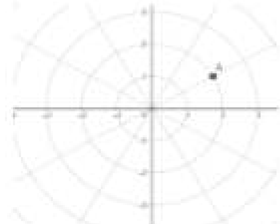
EXERCISE 1.4

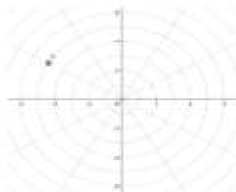
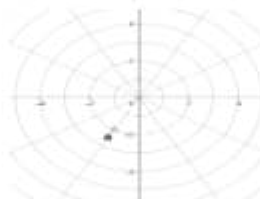
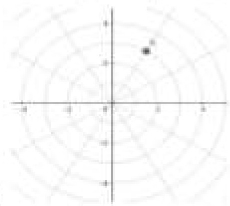
1. i) $\{2, 2\omega, 2\omega^2\}$ ii) $\{-2, -2\omega, -2\omega^2\}$ iii) $\{3, 3\omega, 3\omega^2\}$ iv) $\{-3, -3\omega, -3\omega^2\}$

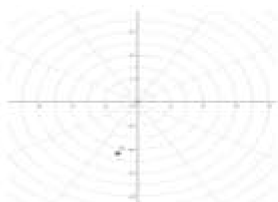
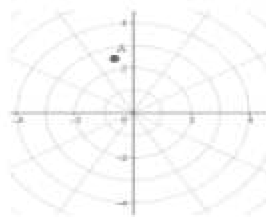
v) $\{4, 4\omega, 4\omega^2\}$ **2.** i) $256\,\omega$ ii) 0 iii) 4 iv) -1 v) -32

6. $x^2+2x+4=0$

EXERCISE 1.5

1. i)  ii)  iii) 

iv)  v)  vi) 

vii)  viii) 

2. i) $5\left(\cos 53.13^\circ + i\sin 53.13^\circ\right)$ ii) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ iii) $1\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

iv) $5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$ **3.** i) $2+2\sqrt{3}i$ ii) $\frac{-3}{4}+\frac{3\sqrt{3}}{4}i$ iii) $-6.47-2.17i$

iv) $-10.69-2.85i$ v) $-2.43+2.86i$ vi) $1.68-1.09i$ vii) $-12+0i$

4. i) $-6.54+15.32i$ (ii) $-1.46+6.68i$ (iii) $45\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$

(iv) $\frac{9}{5}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$ **5.** i) $-1.62+12.47i$ (ii) $-12.69+1.01i$ (iii) $74.64-19.25i$

(iv) $\frac{7}{11}+0i$ **6.** $-1+i\sqrt{3}$ **7.** $-5\sqrt{3}+5i$ **8.** $|z|=2\sqrt{2},\ \arg(z)=\frac{5\pi}{4}+2n\pi$

9. $y=\sqrt{3}x-2\sqrt{3}+1$ **12.** $y=2$ **13.** $x=1$ **14.** $y=\frac{1}{3}$ **15.** $120\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)$

16. Rectangular form: $0+18i$, Polar From: $18\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

EXERCISE 2.1

1. (a) (i) 8 (ii) -1 (iii) x^2-4x+3 (iv) x^4+6x^2+8

(b) (i) $\sqrt{-3}$ (ii) $\sqrt{3}$ (iii) $\sqrt{2x-1}$ (iv) $\sqrt{2x^2+9}$

2. (i) 4 (ii) $\dfrac{2}{h}\cos\left(a+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)$ (iii) $h^2+3ah+h+3a^2+2a$

(iv) $\dfrac{\sinh}{h\cos a\cos(a+h)}$ **3.** (a) $A=\dfrac{P^2}{16}$ (b) $C=2\sqrt{\pi A}$ (c) $S=6V^{2/3}$

4. (i) Domain $g=(-\infty,\infty)$, Range $g=(-\infty,\infty)$

(ii) Domain $g=[-2,\infty)$, Range $g=[0,\infty)$

(iii) Domain $g=(-\infty,\infty)$, Range $g=[0,\infty)$

(iv) Domain $g=(-\infty,\infty)$, Range $g=(-\infty,\infty)$

(v) Domain $g=(-\infty,\infty)$, Range $g=(-\infty,2)\cup[7,\infty)$

5. $a=2$, $b=-2$ **6.** Domain $g=(-\infty,3)\cup(3,\infty)$, Range $g=(-\infty,-1)\cup(-1,\infty)$

7. (i) (a) 30 m (b) 17.5 m (c) 11.1 m (ii) $x=2$ sec

8. (i) Domain $f=(-\infty,\infty)$, Range $f=(-\infty,\infty)$

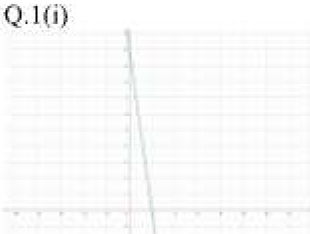
(ii) Yes, the function is one-to-one, because equal ouputs implies equal inputs.

(iii) Yes, the function is onto when the codomain is all real numbers.

9. (i) Domain $f=R-\{-1\}$, Range $f=R-\{2\}$ (ii) $f(x)$ is not onto. 11.$g(x)$ is surjective.

EXERCISE 2.2

Q.1(i)

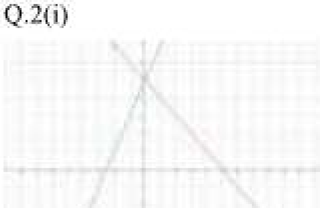


(ii)

(iii)

(iv)

Q.2(i)



(ii)

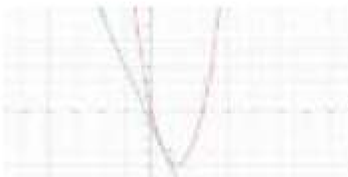
(iii)

(iv)

(v)

(vi)

(vii)



(viii)

3 (i)

(ii)

(iii)

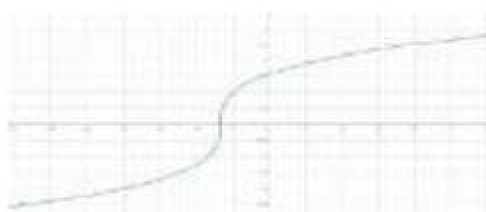
(iv)

(v)

(vi)

(vii)

(viii)



6. (i) (a) 30m (b) 17.5m (c) 11.1m (ii) 2 seconds **7.** (i) 14 months (ii) 373.2 metres **8.** 25 grams

EXERCISE 3.1

1.

(i) Minimum value at $x = -3$ is 4 (ii) Minimum value at $x = -2$ is -4

(iii) Maximum value at $x = 4$ is 29 (iv) Maximum value at $x = \frac{-3}{2}$ is $\frac{-11}{4}$

(v) Minimum value at $x = -1$ is -16 (vi) Maximum value at $x = \frac{-1}{4}$ is $\frac{169}{8}$

2.

(i) Minimum value at $x = 2$ is -4; Domain $f = (-\infty, \infty)$; Range $f = [-4, \infty)$

(ii) Minimum value at $x = \frac{5}{2}$ is $\frac{-1}{4}$; Domain $f = (-\infty, \infty)$; Range $f = [\frac{-1}{4}, \infty)$

(iii) Maximum value at $x = 1$ is -7; Domain $f = (-\infty, \infty)$; Range $f = (-\infty, -7]$

(iv) Minimum value at $x = 2$ is 0; Domain $f = (-\infty, \infty)$; Range $f = [0, \infty)$

(v) Minimum value at $x = -1$ is -9.3; Domain $f = (-\infty, \infty)$; Range $f = [-9.3, \infty)$

(vi) Maximum value at $x = \frac{-1}{2}$ is $\frac{25}{4}$; Domain $f = (-\infty, \infty)$; Range $f = (-\infty, \frac{25}{4}]$

3.

(i) $f^{-1}(x) = -\sqrt{x+3}$; Domain $f^{-1} = [-3, \infty)$; Range $f^{-1} = (-\infty, 0]$

(ii) $f^{-1}(x) = -3 - \sqrt{5+x}$; Domain $f^{-1} = (-5, \infty)$; Range $f^{-1} = (-3, \infty)$

(iii) $f^{-1}(x) = \frac{4 + \sqrt{2\ \ -3 + x}}{2}$; Domain $f^{-1} = [-3, \infty)$; Range $f^{-1} = [2, \infty)$

(iv) $f^{-1}(x) = \frac{1 + \sqrt{3x - 17}}{3}$; Domain $f^{-1} = [71, \infty)$; Range $f^{-1} = [5, \infty)$

(v) $f^{-1}(x) = 3 + \sqrt{\frac{x-1}{2}}$; Domain $f^{-1} = [1, \infty)$; Range $f^{-1} = [3, \infty)$

(vi) $f^{-1}(x) = -4 - \sqrt{\frac{-(x+5)}{3}}$; Domain $f^{-1} = (-\infty, -5)$; Range $f^{-1} = (-\infty, -4]$

4.

(i) $\{-2, 2\}$ (ii) $\{-1, -4\}$ (iii) $\{3 - \sqrt{5}, 3 + \sqrt{5}\}$

(iv) $\left\{\frac{3\ \ \sqrt{7}}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3+\sqrt{7}}{2}\right\}$ (v) $\{(-3, 3)\}$ (vi) $\left\{\left(-\infty, \frac{3-\sqrt{17}}{2}\right) \cup \left(\frac{3+\sqrt{17}}{2}, \infty\right)\right\}$

(vii) $\{(-\sqrt{5}+3, \sqrt{5}+3)\}$ (viii) $\left\{\left(-\frac{3}{2}, \frac{-\sqrt{17}+3}{4}\right) \cup \left(\frac{\sqrt{17}+3}{4}, 3\right)\right\}$

EXERCISE 3.2

1.

(i) $\{1, \frac{1}{2}\}$ (ii) $\{-2, 1\}$ (iii) $\{\frac{-3}{2}, 1\}$ (iv) $\left\{\frac{a+b}{ab}, \frac{2}{a+b}\right\}$

(v) $\{\ \}$ (vi) $\left\{\frac{1}{3}, \frac{-16}{3}\right\}$ (vii) $\{4\}$ (viii) $\{4, 20\}$

(ix) $\{2\}$ (x) $\{4\}$ (xi) $\{0,2\}$ (xii) $\{0, -3\}$

(xiii) $\{2, 4\}$ (xiv) $\{2, 3\}$ (xv) $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$

2. 15 sheep **3.** 97 dozen eggs **4.** 6 hours
5. 20 days **6.** $0 \le s \le 4756$ km/h **7.** [0.586 sec, 3.414 sec]

Exercise 4.1

Q.2 (i) $\begin{bmatrix} -2 & -2 & 3 \\ 2 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ -4 & -2 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} -3 & -2 & 5 \\ 3 & -5 & -3 \\ -3 & -6 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} -1 & -2 & 1 \\ 1 & 5 & -3 \\ 3 & 2 & 2 \end{bmatrix}$

Q.4

(i) $AA' = \begin{bmatrix} 14 & -5 & 8 \\ -5 & 9 & -11 \\ 8 & -11 & 44 \end{bmatrix}$ (ii) $\begin{bmatrix} 11 & -3 & -6 & 1 \\ -3 & 29 & 19 & -5 \\ -6 & 19 & 22 & -4 \\ 1 & -5 & -4 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 2 & 3 & 0 \\ 1 & 0 & 2 & -2 \\ -3 & 5 & 3 & -1 \end{bmatrix}$

Q.5 (i) $X = \begin{bmatrix} 4 & 3 & -2.5 \\ 1 & 3.5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$

Exercise 4.2

1. (i) -21 (ii) -148 (iii) -18 (iv) $9a^2b-b^3$ (v) -123 (vi) $4xyz$

Q.4 (i) $A_{13}=-3, A_{23}=0, A_{33}=7, |A|=-5$

(ii) $B_{31}=-2, B_{32}=-1, B_{33}=2, |B|=-1$

Q.5 (i) $x=2$ or -1 (ii) $x=0$ or 1 (iii) $x=2$ or 3

Q.7(i) 147, 0 (ii) 0, 96

Q.9 $\lambda=\frac{1}{2}, \lambda=-4$

Exercise 4.3

Q.1(i) $\begin{bmatrix} 1 & \frac{17}{4} & \frac{1}{2} \\ 0 & \frac{-1}{2} & 0 \\ \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{-2}{5} & \frac{-2}{5} & \frac{7}{5} \\ \frac{4}{5} & \frac{3}{10} & \frac{-4}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{-1}{5} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{-13}{3} & \frac{8}{3} & \frac{26}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-4}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}$

Q.2(i) Rank = 3 (ii) Rank = 3 (iii) Rank = 4

Q.3(i) $\{(1,0,1)\}$ (ii) $\left\{\left(\frac{68}{55},\frac{59}{55},\frac{62}{55}\right)\right\}$ (iii) $\left\{\left(\frac{8}{3},2,\frac{-7}{3}\right)\right\}$

Q.4(i) $\{(1,1,0)\}$ (ii) $\left\{\left(\frac{-8}{9},\frac{10}{9},\frac{11}{9}\right)\right\}$ (iii) $\{(1,1,1)\}$

Q.5(i) $\left\{\left(\frac{19}{23},\frac{-9}{23},\frac{12}{23}\right)\right\}$ (ii) $\left\{\left(\frac{22}{9},\frac{1}{3},\frac{-10}{9}\right)\right\}$ (iii) $\left\{\left(\frac{61}{16},\frac{-1}{4},\frac{-13}{16}\right)\right\}$

Q.6(i) $\{(0,0,0)\}$ (ii) $x_1=2t, x_2=-t, x_3=t$ for any value of t

(iii) $x_1=-3t, x_2=2t, x_3=t$ for any value of t

7. $A'(-4,1), B'(2,5), C'(0,-3)$ **8.** $A'(-6,-4,1)$ **10.** $\begin{bmatrix}16\\22\\15\end{bmatrix}\begin{bmatrix}36\\43\\49\end{bmatrix}\begin{bmatrix}21\\26\\16\end{bmatrix}$

11. Hold Fire

EXERCISE 5.1

1. $\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$ **2.** $1-\frac{1}{x-1}+\frac{1}{x+1}$ **3.** $\frac{1}{4(x-1)}+\frac{1}{x+2}-\frac{5}{4(x+3)}$

4. $\frac{-1}{28(x-2)}+\frac{30}{7(x+5)}-\frac{5}{4(x+2)}$ **5.** $3x+4+\frac{4}{3(x-1)}+\frac{13}{3(2x+1)}$

6. $1+\frac{3}{8(x-2)}+\frac{3}{4(x-4)}+\frac{15}{8(x-6)}$

7. $\frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x^2+b^2)}+\frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)}+\frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)(x^2+d^2)}$

8. $\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{3}{(x-1)^3}$ **9.** $\frac{5}{x+2}-\frac{22}{(x+2)^2}+\frac{27}{(x+2)^3}$ **10.** $\frac{1}{x-1}-\frac{1}{x+1}+\frac{2}{(x+1)^2}$

11. $2x-2+\frac{162}{25(x-3)}+\frac{288}{25(x+2)}-\frac{32}{5(x+2)^2}$

EXERCISE 5.2

1. $\frac{17x-6}{5(x^2+1)}-\frac{17}{5(x+3)}$ **2.** $\frac{1}{2(1+x)}+\frac{1-x}{2(1+x^2)}$ **3.** $\frac{-2}{13(x+3)}+\frac{2x+33}{13(x^2+4)}$

6. $\frac{2}{3(x+1)}+\frac{x+1}{3(x^2-x+1)}$ **7.** $\frac{1}{4(x-1)}-\frac{x+1}{4(x^2+1)}+\frac{x+1}{2(x^2+1)^2}$

3. $\frac{-1}{36(x-2)}+\frac{x+2}{36(x^2+2)}+\frac{x+14}{6(x^2+2)^2}$

EXERCISE 6.1

1.(i) 24, 28, 32, 36 **(ii)** −3, −5, −7, −9

2.(i) 8, 11, 14 **(ii)** 3, 5, 13 **(iii)** −4, −3, 0 **(iv)** −1, 3, $\frac{3}{5}$

(v) 3, 4, $\frac{14}{3}$ **(vi)** 1, 25, 5929 **(vii)** 4, 16, 36 **(viii)** −7, 28, −63

3. 120 **4.(a)** $6n+1$ **(b)** $10-3n$ **(c)** $\frac{1}{n+1}$ **(d)** $11n-26$

5. 7

EXERCISE 6.2

1.(i) $d=7$; 30,37 **(ii)** $d=\sqrt{2}$; $5+3\sqrt{2}$, $5+4\sqrt{2}$ **2.** **(i)** 2, 15, 28 **(ii)** 12, −1, −14

3. $3n+7$, $4+6n$ **4.(i)** 94 **(ii)** −47 **5.** 75 **6.** No **7.** 5 **8.** 25 **9.** 62 **10.** 7, 12, 17, … ; 502

12. 128 **13.** 164 **14.** $\left(\frac{7n-4}{7}\right)^{10}$; No;Yes **15.** 13

EXERCISE 6.3

1.(i) 2, **(ii)** a^2+b^2 **2.** 1, 21 **3.** $\frac{25}{6\sqrt{2}},\frac{19}{3\sqrt{2}},\frac{17}{2\sqrt{2}},\frac{32}{3\sqrt{2}},\frac{77}{6\sqrt{2}}$ **4.** 5, 9 or 9,5 7. 0

EXERCISE 6.4

1.(i) 630 (ii) $\frac{n(n+7)}{2\sqrt{5}}$ 2.(i) 1300 (ii) 230 (iii) 1932 3. 22 4. 14, 51

5. 9*cm*, 12*cm*, 15*cm* 6. (i) $n(3n-2)$ (ii) $\frac{n}{2}(9n-13)$ 7. 650 8. 385

9. 200000 10. 3 + 7 + 11+ … 11. 73 12. 5,8, 11 or 11, 8, 5 13. 32

14. 5, 7, 9 , 11 or 11, 9, 7, 5 15. 3, 4, 5, 6, 7 or 7, 6, 5, 4, 3 17. 11

EXERCISE 6.5

1. $\frac{-3}{16}$ 2. 6561 3. 5 4.(i) 243, 81, 27, 9, 3 (ii) 579, $\frac{-579}{2},\frac{579}{4},\frac{-579}{8},\frac{579}{16}$

5. −64 6. 2, 6, 18, … ; $2\cdot 3^{n-1}$ 8. $\sqrt{mn}$ 9. 2, 6, 18 or 18,6, 2 10. $81\cdot\sqrt[3]{9}$

12. 2, 7, 12 or 10, 7, 4 13. 1, 2, 3 or 17, 2, −13 15. $\frac{-81}{4}$

EXERCISE 6.6

1. (i) $4i$ or $-4i$ (ii) 4 or −4 (iii) $3\sqrt{6}$ or $-3\sqrt{6}$ 2. 6, 12, 24, 48 **4.** $\frac{1}{2}$ **5.** 4, 16 or 16,4

6. 2,8 or 8, 2

EXERCISE 6.7

1. $\frac{7174453}{4782969}$ 2. 4, 172 3.(i) $\frac{2}{9}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$ (ii) $\frac{1}{3}\left[\frac{10}{9}\left(10^n-1\right)-n\right]$

4.(i) $\frac{a(1-b)(1-a^n)-b(1-a)\left(1-b^n\right)}{(a-b)(1-a)(1-b)}$ (ii) $\frac{r}{1-k}\left\{\frac{1-r^n}{1-r}-\frac{k\left(1-k^n r^n\right)}{1-kr}\right\}$

5. $\frac{15(1-i)}{8}$

EXERCISE 6.8

1. 14080 2. $2(4n-1)\cdot 3^{n-1}$ 3. $(2n+3)(-3)^n$; −195 4.(i) $6+(4n-6)\cdot 2^n$

(ii) $\frac{3}{2}[1-(n+1)3^n+n\cdot 3^{n+1}]$ (iii) $4-\frac{4}{3}\left(\frac{1}{4}\right)^{n-1}-\frac{4}{3}(3n-1)\left(\frac{1}{4}\right)^n$

(iv) $\frac{15}{8}-\frac{5}{4}(2n-1)\left(\frac{1}{5}\right)^n-\frac{5}{8}\left(\frac{1}{5}\right)^{n-1}$ (v) $\frac{15}{4}-\frac{3}{2}(3n-2)\left(\frac{1}{3}\right)^n-\frac{9}{4}\left(\frac{1}{3}\right)^{n-1}$

5. (i) 6 (ii) $\frac{21}{4}$ 8. $\frac{2-(n+1)x^n+2nx^{n+1}}{(1-x)^2}$ 9. $n(2n+1)$

11. $\frac{2}{1-x}+\frac{3x(1-x^{n-1})}{(1-x)^2}-\frac{(3n-1)x^n}{1-x}$; $\frac{2+x}{(1-x)^2}$

EXERCISE 6.9

1.(i) $\frac{1}{19}$ **(ii)** $\frac{1}{11}$ **2.(i)** $-1, 2, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}$ **(ii)** $\frac{3}{22}, \frac{3}{32}, \frac{1}{14}, \frac{3}{52}, \frac{3}{62}$ **3.** $\frac{1}{13}$

4. −10 **5.** 67 **6.** −1 **8.** 3, 6 or 6, 3 **9.** 2, 8 or 8, 2

EXERCISE 6.10

1.(i) $\frac{n}{2}\left(2n^2+n-1\right)$ **(ii)** $\frac{n(n+1)(4n-1)}{2}$ **(iii)** $n(n+1)^2$

(iv) $\frac{n}{3}(8n^2+21n+16)$ (v) $\frac{n}{3}\left(4n^2-1\right)$ (vi) $\frac{n^2(n+1)^2}{2}$

(vii) $\frac{n}{6}\left(3n^3+16n^2+30n+23\right)$ (viii) $\frac{n(n+1)(n+2)}{6}$

(ix) $\frac{1}{12}n(n+1)(n^2+3n+2)$ (x) $\frac{n}{6}\left(9n^3+58n^2+135n+134\right)$

2. (i) $-n(2n+1)$ (ii) $\frac{n}{36}\left(4n^2+15n+17\right)$ 3.(i) $n\left(n^2+2n+2\right)$ (ii) $\frac{n}{6}\left(2n^2+15n+19\right)$

4.(i) $n\left(8n^2+10n+5\right)$ (ii) $n\left(4n^3+4n^2+5n+8\right)$

EXERCISE 6.11

1. Rs. 65 **2.** Rs. 239077.50 **3.** 5% **4.** Rs. 173596

5. (a) 900 litres, **(b)** 200 weeks, **(c)** 400 weeks **6. (a)** 23.8 million, **(b)** $7+1.4n$,

(c) 21 **7. (a)** 100, 80, 64, 51.2, … **(b)** 482.4 **(c)** 500 **8.** Rs. 8000

9. Rs. 9468.22 **10.** 17 hours **11.** 25 days **12.** 1088
13. 7.2 seconds 14. 410.4mA

EXERCISE 7.1

1. 12 kinds of rolls **2.** 12 career paths **3.** i) 5040 ii) 362,880 iii) 90 iv) 1320 v) 36 vi) 10 vii) 25,200 viii) 110,880 ix) 220 x) 1 xi) 40,320 xii) 1440

4. i) $\frac{8!}{4!}$ ii) $\frac{15!}{10!}$ iii) $\frac{19!}{15!}$ iv) $\frac{11!\cdot 3!}{5!\cdot 6!}$ v) $\frac{10!}{5!\cdot 5!}$ vi) $\frac{50!}{5!\cdot 46!}$ vii) $\frac{n!}{(n-4)!}$

viii) $\frac{(n+2)!}{(n-2)!}$ ix) $\frac{(n+3)!}{(n-1)!\cdot 5!}$ x) $\frac{n!}{(n-r+1)!}$

EXERCISE 7.2

1. i) 30,240 ii) 20 iii) 5040 iv) 720 **2.** i) 9 ii) 5 iii) 10 **4.** 30
5. i) 6,227,020,800 ii) 51,891,840 iii) 1,037,836,800 **6.** 5040 **7.** **(**a) 1440 (b) 35,280
8. 665,280 **9.** a) 3,628,800 b) 3,386,880 **10. a)** 6,227,020,800 b) 622, 080 c) 239,500,800
11. 120 **12.** 30240 **13.** 1440 **14.** 2880

EXERCISE 7.3

1. i) 151,200 ii) 479,001,600 iii) 9,979,200 iv) 10810800 **2.** 1260
3. **a)** case-I: 5040, case-II: 2520 **b)** case-I: 720, case-II: 360 **c)** case-I: 120, case-II: 60 **4.** 2880
5. 180 **6.** 360 **7.** 12,612,600 **8.** 725,760 **9.** 6,227,020,800 ways **10.** 967680
11. 2880 **12.** 3 **13.** 60

EXERCISE 7.4

1. i) 10 ii) 56 iii) 1 iv) 120 **2.** i) 8 ii) 14 iii) 15 **3.** 56 **4.** 65,780
5. 560 **6.** 171,028,000 **7.** i) 1176 ii) 280 iii) 490 iv) 56 **8.** i) 10 ii) 20 iii) 54
9. 1176 **10.** 20 **11.** 1365, 1001 **13**. (i) 840 (ii) 1016 iii) 1008 **15**. (i) 358,800
(ii) 14,950 **16.** (i) 86,400 ii) 120 **17.** (i) $\frac{1}{2730}$ ii) $\frac{1}{455}$ **18.** (i) 518,400 ii) 14,400

EXERCISE 8.2

1. (i) $\frac{x^6}{64}-\frac{3}{8}x^3+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$

(ii) $128a^7-448a^5x+672a^3x^2-560ax^3+280\frac{x^4}{a}-84\frac{x^5}{a^3}+14\frac{x^6}{a^5}-\frac{x^7}{a^7}$

(ii) $\frac{a^3}{x^3}-\frac{6a^2}{x^2}+\frac{15a}{x}-20+\frac{15x}{a}-\frac{6x^2}{a^2}+\frac{x^3}{a^3}$ **2.** (i) 0.91267 (ii) 16.64966416 (iii) 9920.23968016 (iv) 40.84101 **3.** (i) $2a^4+20a^2x^2+8x^4$ (ii) 724

4. (i) $16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$ (ii) $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^6 + 10x^6 - 4x^7 + x^8$ **5.** (i) $15120x^4$ (ii) $-41184x^{-2}$ (iii) $4032\dfrac{a^4}{x^5}$ (iv) $462\, x^5y$ **6.** (i) $\dfrac{-15309}{8}$

(ii) $\dfrac{(-1)^n(2n)!}{(n!)^2}$ **7.** $\dfrac{-15309}{8}x^5$ **8.** (i) -8064 (ii) $\dfrac{45}{4}$ (iii) 35

EXERCISE 8.3

1. (i) $1-\dfrac{1}{3}x+\dfrac{2}{9}x^2-\dfrac{14}{81}x^3+\ldots$ is valid if $|x|<1$ (ii) $2-\dfrac{2}{4}x-\dfrac{9}{64}x^2-\dfrac{27}{512}x^3+\ldots$ is valid if $\left|\dfrac{3}{4}x\right|<1 \Rightarrow |x|<\dfrac{4}{3}$ (iii) $1 - x + 2x^2 - 2x^3 + \ldots$ is valid if $|x|<1$ (iv) $1 + 2x + \dfrac{3}{2}x^2 + 2x^3 + \ldots$ is valid if $|x|<\dfrac{1}{2}$ **2.** (i) 9.950 approximate (Correct to three decimal places) (ii) 1.010 approximate (Correct to three decimal places) (iii) 0.331 approximate (Correct to three decimal places) (iv) 0.935 approximate (Correct to three decimal places)

3. (i) $(-1)^n \times 2n$ (ii) $4n$ **7.** $\dfrac{2}{\sqrt{5}}$

EXERCISE 8.4

6. (i) 0.3679 (Approximately) (ii) 0.000045 **7.** 56 **8.** Rs. 12,616,000 9. 63 items **10.** Rs. 2,928,200 **11.** 28 matches **13.** 180,160 items

EXERCISE 9.1

1. (i) Quotient = $3x + 2$, Remainder = 4 (ii) Quotient = $x^2 + 14x + 25$, Remainder = 54 (iii) Quotient = $x^3 + x^2 - 2x + 1$, Remainder = 18 (iv) Quotient = $5x^2 - 3x - 18$, Remainder = $12x + 71$ (v) Quotient = $3x^2 + 4x - 3$, Remainder = $-25x + 9$ **2.** (i) 20 (ii) 10 (iii) 5 (iv) 91 (v) 10
3. (i) $x + 1$ is a factor of $x^2 - 1$ (ii) $x - 2$ is a factor of $x^2 - 5x + 6$
(iii) $x + 1$ is not a factor of $x^3 + x^2 + x - 3$ (iv) $x - 2$ is a factor of $x^3 + x^2 - 7x + 2$
(iv) $x - 3$ is not a factor of $x^4 - 3x^3 + x^2 - x + 1$
4. (i) $(x-2)(x-1)(x+3)$ (ii) $(x+4)(x-6)(x+2)$
(iii) $(x-2)(x+3)(x+1)(2x+3)$
5. Quotient = $x^3 - 3x^2 - x + 1$, Remainder = 1 **6.** p = 2, q = −1 **7.** k = 1 **8.** k= 8

9. $p=\dfrac{-5}{2}, q=\dfrac{-1}{2}$ **10.** $a = -8$, $b = -16$

Exercise 9.2

1. 26.25% **2.** $x = -1$ is a valid point **3.** $x = 2$ lies on the curve
4. $x + 1$ is not a factor of p(x) **5.** CRC = 20 **6.** (i) Remainder = 1
(ii) System response in not zero when $x = 1$ **7.** 45
8. System response in not zero when $t = 4$ **9.** Received message is not error-free, because remainder is non-zero. **10.** Code word is not valid, because $x - 1$is not a factor of $C(x)$.

EXERCISE 10.1

1. i) $\frac{\sqrt{3}}{2}$ ii) -1 iii) 2 iv) -2 v) $\frac{1}{\sqrt{3}}$ vi) $\frac{-1}{2}$ **2.** i) $-\cos 12^o$ ii) $-\sin 12^o$ iii) $\cos 27^o$

iv) $\tan 33^o$ v) $\sin 15^o$ vi) $-\sin 39^o$ vii) $-\cot 33^o$ viii) $-\sin 21^o$ ix) $-\sin 30^\circ$

Exercise 10.2

2. i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ ii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ iii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ iv) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ v) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ vi) $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

9. i) $-\frac{56}{65}$ ii) $-\frac{33}{65}$ iii) $\frac{56}{33}$ iv) $\frac{16}{65}$ v) $\frac{63}{65}$ vi) $\frac{16}{63}$

The terminal arms of angles of measure and $\alpha+\beta$ and $\alpha-\beta$ are in III and I quadrants respectively.

10. i) $\frac{33}{65}, -\frac{56}{65}$ ii) $\frac{416}{425}, \frac{3}{5}$ **14.** i) $13\sin(\alpha+\phi)$, $\tan\phi = \frac{5}{12}$ ii) $5\sin(\theta+\varphi)$, $\tan\varphi = \frac{4}{3}$

$\sqrt{2}\sin(\theta+\phi), \tan\phi = -1$ iv) $\sqrt{41}\sin(\theta+\varphi)$, $\tan\varphi = -\frac{4}{5}$ v) $\sqrt{2}\sin(\theta+\varphi)$, tan $\varphi = 1$ vi) $\sqrt{34}\sin(\theta+\phi), \tan\phi = \frac{-5}{3}$

EXERCISE 10.3

1. i) $\sin 2\alpha = \frac{120}{169}$, $\cos 2\alpha = -\frac{119}{169}$, $\tan 2\alpha = -\frac{120}{119}$ ii) $\sin 2\alpha = \frac{24}{25}$, $\cos 2\alpha = -\frac{7}{25}$, $\tan 2\alpha = -\frac{24}{7}$

14. $\sin^4\theta = \frac{3-4\cos 2\theta+\cos 4\theta}{8}$ **15.** i) $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$ ii) $\sin 54^\circ = \frac{\sqrt{5}+1}{4} = \cos 36^\circ$ iii) $\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^\circ$ iv) $\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} = \sin 36^\circ$

EXERCISE 10.4

1. i) $\sin 4\theta + \sin 2\theta$ ii) $\sin 8\theta - \sin 2\theta$ iii) $\frac{1}{2}(\sin 7\theta + \sin 3\theta)$ iv) $\cos 5\theta - \cos 9\theta$

v) $\frac{1}{2}(\sin 2x - \sin 2y)$ vi) $\frac{1}{2}(\cos 4x + \cos 60^o)$ vii) $\frac{1}{2}(\cos 34^o - \cos 58^o)$ viii) $\frac{1}{2}(\cos 90^o - \cos 2x)$

2. i) $2\sin 4\theta\cos\theta$ ii) $2\cos 6\theta\sin 2\theta$ iii) $2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$ iv) $-2\sin 4\theta\sin 3\theta$

v) $2\cos 30^o\cos 18^o$ vi) $2\sin x\cos 30^o$

Exercise 11.1

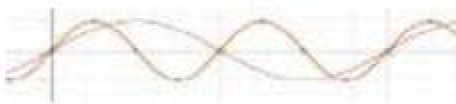
1. (i) even (ii) neither even nor odd (iii) even (iv) neither even nor odd (v) odd (vi) odd (vii) even (viii) even **2.** (i) $\frac{2\pi}{3}$ (ii) $\frac{2\pi}{7}$ (iii) $\frac{\pi}{3}$ (iv) 2π (v) $\frac{40}{\pi}$ (vi) 5π (vii) $\frac{4\pi}{3}$ (viii) $\frac{2}{7}$ (ix) 30 (x) $\frac{4\pi}{7}$ (xi) 30π

Exercise 11.2

1. (i) ii)

(iii)  **(iv)** 

(v)  **(vi)** 

2. **(i)**  **(ii)** 

Exercise 11.3

1. (i) Max=4, Min=2 (ii) Max=4, Min=2 (iii) Max=$\frac{3}{2}$, Min=$\frac{-1}{2}$ (iv) Max=$\frac{5}{2}$, Min=$\frac{1}{2}$ (v) Max=4, Min=−2 (vi) Max=3, Min=−1 (vii) Max=$\frac{1}{8}$, Min= $\frac{1}{12}$ (viii) Max=$\frac{1}{4}$, Min=$\frac{1}{10}$ (ix) Max=$\frac{1}{2}$, Min= $\frac{1}{8}$ **2.** (a) maximum temperature= 21.5°, minimum temperature= 8.5° (b) Temperature at 9 AM = 8.89° **3.** distance=36.78m **4.** height=30.92m

5. (a) $h(t) = -30\cos\left(\frac{\pi}{40}t\right) + 36$ (b) 66 feet (c) 63.72 feet **6.** (a) 2.7 m (b) 0.3m (c) $\frac{2}{3}$ second (d) 0.05 second **7.** (a) $h(t) = 28 - 20\cos\left(\frac{\pi}{60}t\right)$ (b) 28 feet (c) 37.87s and 82.13s **8.** (a) $69.66^{\circ}F$ (b) 6 hr (c) $72^{\circ}F$ **9.** (a) 65000 (b) 80000

EXERCISE 12.1

1. (i) 2 (ii) 0 (iii) $\frac{5}{2}$ (iv) $\frac{1}{2}$ **.2** (i) 10 (ii) 5 (iii) 4 (iv) 0 (v) 0 (vi) $\frac{13}{4}$

3. (i) 2 (ii) 4 (iii) $\frac{12}{5}$ (iv) 0 (v) $-\frac{1}{2}$ (vi) 1 (vii) $\frac{1}{2\sqrt{2}}$ (viii) $\frac{1}{2\sqrt{x}}$

(ix) $\frac{n}{m}a^{n-m}$ **4.** (i) 5 (ii) $\frac{\pi}{180}$ (iii) 0 (iv) 1 (v) $\frac{a}{b}$ (vi) 1 (vii) 2 (ix) 0

(x) 1 (xi) $\frac{3}{2}$ (xii) $-\frac{1}{2}$ **5.** (i) e^2 (ii) $\sqrt{e}$ (iii) $\frac{1}{e}$ (iv) $e^{\frac{1}{3}}$ (v) e^4 (vi) e^6

(vii) e^2 (viii) $\frac{1}{e^2}$ (ix) $\frac{1}{e}$ (x) -1 (xi) 1

EXERCISE 12.2

1. (i) -2 (ii) 0 (iii) 0 **2.** (i) f is discontinuous at $x = 2$ (ii) f is discontinuous at $x = 1$
3. (i) f is discontinuous at $x = 2$ (ii) f is discontinuous at $x = -2$ **4.** c = -1

5. (i) $m = 1, n = 3$ (ii) $m = 4$ **6.** $k = \frac{1}{6}$ **7.** f(x) is discontinuous at $x = 1$

EXERCISE 12.3

1. 0 **2.** 100,00010 **3.** 500 **4.** (i) 10 (ii) 0 **5.** (i) ∞ (ii) 82.44
6. yes **7.** (i) 16.18% (ii) 134.99 **8.** yes

3. (i) 2 (ii) 4 (iii) $\frac{12}{5}$ (iv) 0 (v) $-\frac{1}{2}$ (vi) 1 (vii) $\frac{1}{2\sqrt{2}}$ (viii) $\frac{1}{2\sqrt{x}}$

(ix) $\frac{n}{m}a^{n-m}$ **4.** (i) 5 (ii) $\frac{\pi}{180}$ (iii) 0 (iv) 1 (v) $\frac{a}{b}$ (vi) 1 (vii) 2 (ix) 0

EXERCISE 12.4

1. 10% **2.** 15% **3.** 8% **4.** 1400 **5.** 18000 **6.** Year 1 = 4800, year 2= 3600
7. depreciable cost = 90000 , year 2 = 24000 **8.** 2250 **9.** 2667 **10.** 67500
11. Year 1 = 32000, Year 2 = 22400

EXERCISE 13.1

1. (i) $4x$ (ii) $\frac{-1}{2\sqrt{x}}$ (iii) $-\frac{1}{2}x^{-3/2}$ (iv) $2x - 3$ **2.** (i) $\frac{1}{4\sqrt{2}}$ (ii) $-\frac{1}{4\sqrt{2}a^{3/2}}$

3. (i) $\frac{1}{3}$ (ii) $2x + 2$ **4.** (i) $\frac{-6}{(3x-2)^3}$ (ii) $10(2x+3)^4$ (iii) $7a(ax+b)^6$

5. 8, $y = 8x + 13$ **6.** $-5, y = -5x - 87$. 1, $y = -5x - 8$ **8.** 8 **9.** $\frac{1}{6}$, $6y = x + 9$

10. (a) 28*km/h* (b) $\frac{13}{3}$*km/h* **11.** −2*ft/sec* **12.** 8°c/*hr* **13.** (i) not differentiable (ii) not differentiable

EXERCISE 13.2

1. (i) $4x^3+6x^2+2x$ (ii) $-3\left(\frac{1}{x^4}+\frac{1}{x^{5/2}}\right)$ (iii) $\frac{8}{(2x+1)^2}$ (iv) $\frac{1-3x}{2\sqrt{x}}$ (v) $1-2x^{-3}+x^{-3/2}$

(vi) $8-2x$ (vii) $\frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}$ (vii) $\frac{-8x}{(x^2-3)^2}$ (ix) $\frac{x+2}{(x^2+1)^{3/2}}$

(x) $\frac{-a}{\sqrt{a-x}\,(a+x)^{3/2}}$ (xi) $\frac{-2x}{\sqrt{x^2+1}\,(x^2-1)^{3/2}}$ **2.** $\frac{3x^2-2x^{3/2}-3x+2}{2\sqrt{x}\,(\sqrt{x}-1)^2}$ **3.** $\frac{x^3-3x^2+3x-1}{2\sqrt{x}\,(x^{3/2}-x^{1/2})^2}$

EXERCISE 13.3

4. $v=15t^2-6t+1$ **5.** Max. stress = 100, Rate of change = 0

6. (a) $P(x)=-10x^2+700x-2000$ (b) Rs. 400 (c) 35 units

8. (a) 2940 (b) 27440 (c) as time increases rate increases

11. (a) 152*m/s* (b) 96*m/s*^{2} (c) $t=0.47$ sec and $t=1.13$ sec **12.** (a) 72*km/h* (b) −12*km/h*^{2} (c) 2.5*hrs* **13.** (a) 292*Pa/m* (b) x = 11.55m (c) increasing **14.** (a) $r=1.44m$ (b) Rs. 156250.57 (c) 191686.6 units/*m*

EXERCISE 14.1

1. **(i)** $\underline{i}-9\underline{j}$ **(ii)** $13\underline{i}-2\underline{j}-22\underline{k}$ **(iii)** $\sqrt{273}$

2.(i) $7; \frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$ **(ii)** $6; \frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$ **(iii)** $10; \frac{-3}{5}, \frac{4}{5}, 0$ **3.** $t=\frac{1\pm\sqrt{17}}{2}$

4. $-\frac{1}{9}\underline{i}+\frac{4}{9}\underline{j}-\frac{8}{9}\underline{k}$ **5.** $\frac{17\underline{i}-12\underline{j}-16\underline{k}}{\sqrt{689}}$ **6.(i)** $\frac{15}{\sqrt{26}}\underline{i}+\frac{20}{\sqrt{26}}\underline{j}-\frac{5}{\sqrt{26}}\underline{k}$

(ii) $-\frac{7}{\sqrt{3}}\underline{i}+\frac{7}{\sqrt{3}}\underline{j}+\frac{7}{\sqrt{3}}\underline{k}$ **7.** $x=-3, y=-5$

9.(a) $\frac{2}{3}\underline{i}-\frac{4}{3}\underline{j}+\frac{4}{3}\underline{k}$ and $-\frac{2}{3}\underline{i}+\frac{4}{3}\underline{j}-\frac{4}{3}\underline{k}$ **(b)** −3 **(c)** $\frac{-5\underline{i}+10\underline{j}-15k}{\sqrt{14}}$

(d) $a=-\frac{3}{2}, b=\frac{1}{2}$ **10.** $10\sqrt{179}$ kilometers **11.(i)** $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$ **(ii)** $\frac{4}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}, -\frac{\sqrt{5}}{3}$

12. Only the triple (iii) 45°, 60°, 60° satisfies the condition for direction angles of a single vector.

EXERCISE 14.2

1.(i) $\frac{\sqrt{14}}{7}$ **(ii)** $\frac{9}{\sqrt{870}}$ **(iii)** $\frac{-1}{\sqrt{1558}}$ **(iv)** $\frac{-1}{\sqrt{6}}$

2.(i) Projection of $\underline{a}$ along $\underline{b}$: $-\frac{8}{21}\underline{i}+\frac{16}{21}\underline{j}-\frac{32}{21}\underline{k}$; Projection of $\underline{b}$ along $\underline{a}$: $-\frac{8}{7}\underline{i}-\frac{12}{7}\underline{j}+\frac{4}{7}\underline{k}$

(ii) Projection of $\underset{\sim}{a}$ along $\underset{\sim}{b}$: $\frac{5}{3}\underset{\sim}{i}+\frac{5}{3}\underset{\sim}{j}+\frac{5}{3}\underset{\sim}{k}$; Projection of $\underset{\sim}{b}$ along $\underset{\sim}{a}$: $\frac{20}{29}\underset{\sim}{i}-\frac{10}{29}\underset{\sim}{j}+\frac{15}{29}\underset{\sim}{k}$

3.(i) 3 **(ii)** 1 or $-\frac{3}{2}$ **4.** 2 or −3 **5.** zero vector

6.(ii) The points $P(4, -1, 2)$, $Q(1, 3, -1)$, $R(-2, 4, 6)$ do not form a right triangle.

9. 56 Nm **10.** 32 Nm **12.** $\frac{99\sqrt{26}}{13}\,Nm$

EXERCISE 14.3

1.(i) $\underset{\sim}{a}\times\underset{\sim}{b}=-3\underset{\sim}{j}-3k$; $\underset{\sim}{b}\times\underset{\sim}{a}=3\underset{\sim}{j}+3\underset{\sim}{k}$ **(ii)** $\underset{\sim}{a}\times\underset{\sim}{b}=5\underset{\sim}{i}+3\underset{\sim}{j}-7\underset{\sim}{k}$; $\underset{\sim}{b}\times\underset{\sim}{a}=-5\underset{\sim}{i}-3\underset{\sim}{j}+7\underset{\sim}{k}$

(iii) $\underset{\sim}{a}\times\underset{\sim}{b}=-7\underset{\sim}{i}-7\underset{\sim}{j}$; $\underset{\sim}{b}\times\underset{\sim}{a}=7\underset{\sim}{i}+7\underset{\sim}{j}$ **(iv)** $\underset{\sim}{a}\times\underset{\sim}{b}=3\underset{\sim}{i}-6\underset{\sim}{k}$; $\underset{\sim}{b}\times\underset{\sim}{a}=-3\underset{\sim}{i}+6\underset{\sim}{k}$

2.(i) $\frac{21\underset{\sim}{i}-9\underset{\sim}{j}-11\underset{\sim}{k}}{\sqrt{643}}$; $\sin\theta=\frac{\sqrt{643}}{\sqrt{644}}$ **(ii)** $\frac{-7\underset{\sim}{i}+2\underset{\sim}{j}+5\underset{\sim}{k}}{\sqrt{78}}$; $\sin\theta=\frac{\sqrt{78}}{\sqrt{87}}$ (iii) $\frac{\underset{\sim}{j}-\underset{\sim}{k}}{\sqrt{2}}$; $\sin\theta=\frac{2\sqrt{2}}{3}$

(iv) $\frac{13\underset{\sim}{i}+\underset{\sim}{j}+22k}{\sqrt{654}}$; $\sin\theta=\frac{\sqrt{654}}{\sqrt{735}}$ **3.(i)** $\frac{3\sqrt{26}}{2}$ square units **(ii)** $\frac{5\sqrt{2}}{2}$ square units

4(i) $5\sqrt{3}$ square units **(ii)** $\sqrt{237}$ square units **(iii)** $\sqrt{190}$ square units

5. $a=\frac{21}{5}$, $b=\frac{12}{5}$ **6.(i)** Parallel vectors: $\underset{\sim}{u}$ and $\underset{\sim}{w}$; Perpendicular vectors: No

(ii) Parallel vectors: $\underset{\sim}{u}$ and $\underset{\sim}{w}$; Perpendicular vectors: $\underset{\sim}{u}$ and $\underset{\sim}{v}$; $\underset{\sim}{v}$ and $\underset{\sim}{w}$

11. Conclusion: At least one of the vectors $\underset{\sim}{a}$ or $\underset{\sim}{b}$ is the zero vector.

13. $48\underset{\sim}{i}-4\underset{\sim}{j}+30\underset{\sim}{k}$ **14.** $-14\underset{\sim}{j}-14\underset{\sim}{k}$ **15.** $3\underset{\sim}{i}+3\underset{\sim}{j}+3\underset{\sim}{k}$ **16.** $15\underset{\sim}{i}-15\underset{\sim}{j}-15\underset{\sim}{k}$

EXERCISE 14.4

1.(i) 25 cubic units **(ii)** 14 cubic units **(iii)** 10 cubic units **4.(i)** $\frac{5}{2}$ **(ii)** ±1

6.(a)(i) 4 **(ii)** 3 **(iii)** 1 **(iv)** 0 **7.(i)** $\frac{8}{3}$ cubic units **(ii)** $\frac{2}{3}$ cubic units

10. $\frac{301}{\sqrt{1630}}$ **11.** $150\underset{\sim}{j}-100\underset{\sim}{k}$ (in pound feet) **12.** $\sqrt{41}$ meters

13. Rs. 532500, which is the total revenue from the sales of all items.

14. $-20\underset{\sim}{i}+110\underset{\sim}{j}+50\underset{\sim}{k}\,Nm$ **15.(a)** [500, 300, 200], [500, 400, 2000] **(b)** Rs. 770000

Glossary

Complex Numbers: The numbers of the form $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, are called complex numbers.

Conjugate Complex Numbers: Let $z = a + ib$ be a *complex number,* then $a - ib$ is called the complex conjugate of $a + ib$. **Complex polynomial**: **Complex polynomial** $P(z)$ is a polynomial function of the complex variable z with complex coefficients. It is expressed in the general form as: $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$.

Zeros of the function: If $P(z)$ is a polynomial function, the values of z that satisfy $P(z) = 0$ are called the zeros (or roots) of the function.

Imaginary cube roots of unity: The numbers containing i are called Complex numbers. So $\frac{1+\sqrt{3}i}{2}$ and $\frac{1\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

Elements of the matrix: The numbers used in rows or columns are said to be the **entries** or **elements** of the matrix.

Order of matrix: A bracketed rectangular array of $m \times n$ elements $a_{ij}(i = 1, 2, 3, ..., m;\ j = 1, 2, 3, ..., n)$, arranged in m rows and n columns is called an m by n matrix (written as $m \times n$ matrix), where $m \times n$ is called the *order* of the matrix.

Row Matrix or Row vector: A matrix, which has only one row, i.e., $1 \times n$ matrix of the form $\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & ... & a_{in} \end{bmatrix}$ is said to be a row matrix or a row vector.

Rectangular Matrix: If $m \neq n$, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular matrix.

Square Matrix: If $m = n$, then the matrix of order $m \times n$ is said to be a square matrix of order n or m. i.e., the matrix which has the same number of rows and columns is called a square matrix.

Null Matrix or Zero Matrix: A square or rectangular matrix whose each element is zero, is called a *null or zero matrix*.

Transpose of a Matrix: If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A. It is denoted by A'.

Inverse of a Square Matrix of Order $n \geq 3$: Let A be a non-singular square matrix of order n. If there exists matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}.

Partial Fraction: Expressing a rational function as a sum of partial fractions is called Partial Fraction.

Rational Fraction: The quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, with no common factors, is called a Rational Fraction.

Proper Rational Fraction: A rational function $\frac{P(x)}{Q(x)}$ is called a Proper Rational Fraction if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator.

Improper Rational Fraction: A rational fraction $\frac{P(x)}{Q(x)}$ is called an Improper Rational Fraction if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator.

Irreducible Factor: A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, $x^2 + x + 1$ and $x^2 + 3$ are irreducible quadratic factors.

Fundamental Law of Trigonometry: Let a and b be any two angles (real numbers), then $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ which is called the Fundamental Law of Trigonometry.

Allied Angles: The angles associated with basic angles of measure θ to a right angle or its multiple are called Allied Angles.

Function: A function is a rule or correspondence, relating two sets in such a way that each element in the first set corresponds to one and only one element in the second set.

Domain: A function *f* from a set *X* to a set *Y* is a rule or a correspondence that assigns to each element *x* in *X* a unique element *y* in *Y*. The set *X* is called the domain of *f*.

Range: The set of corresponding elements *y* in *Y* is called the range of *f*.

Even Function: A function f is said to be an even if $f(-x)=f(x)$, for every number x in the domain of f.

Odd Function: A function f is said to be an odd if $f(-x)=-f(x)$, for every number x in the domain of f.

Vector: A vector is a quantity that has both magnitude and direction for examples displacement, velocity, acceleration, weight, force, momentum, electric and magnetic fields, etc.

Scalar: A scalar is a quantity that has only magnitude or size, such as mass, time, density, temperature, length, volume, speed work etc.

Unit Vector: A unit vector is defined as a vector whose magnitude is unity.

Orthogonality of Two Vectors: Two non-zero vectors $\underline{u}$ and $\underline{v}$ are perpendicular if and only if $\underline{u}\times\underline{v}=0$.

Hypothesis: A hypothesis is an educated guess or proposed explanation for a statement based on limited evidence.

Induction of Hypothesis: It refers to the process of formulating a general statement or hypothesis based on specific examples or patterns observed in particular cases.

Binomial Expression: An algebraic expression consisting of two terms such as $a+x$, $x-2y$, $ax+b$ etc., is called a binomial or a binomial expression.

Factorial: Factorial is a mathematical operation that multiply a number by every positive integer below it till 1.

Permutation: A permutation of n different objects taken $r(\leq n)$ at a time is an arrangement of the r objects.

Circular Permutation: In circular permutation, there are $(n-1)!$ ways for n distinct things or objects because in circular order, arrangements of things / objects can be rotated $(n-1)!$ times.

Limit of a Function: Let a function $f(x)$ be defined in an open interval near the number "a" (need not to be at "a". If, as x approaches "a" from both left and right sides of "a", $f(x)$ approaches a specific number "L". Then "L" is called the limit of $f(x)$ as x approaches to a.

Divergent Sequences: A sequence is divergent if it does not approach a finite value.

Monotonic Sequences: A sequence is monotonic if it is either entirely non-increasing or non-decreasing. Monotonic sequences often converge, but not always.

Bounded Sequences: A sequence is **bounded** if there exists some real number M such that |an| ≤ M| for all n. A bounded sequence may or may not converge.

Arithmetic progression (A.P): An arithmetic progression is a sequence in which each term after the first is found by adding a constant to the previous term. This constant is called common difference of the arithmetic progression and is usually denoted by'd'.

Series: The sum of the terms of a sequence is called the series of the corresponding sequence.

Geometric Progression (G.P): A geometric progression or geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a nonzero constant r called common ratio.

Arithmetic geometric sequence (A.G.S): A sequence which is formed by multiplying the corresponding terms of an A.P. and a G.P. is called arithmetic-geometric sequence.

Quadratic function: A quadratic function is a polynomial function of degree two. It is typically expressed in the standard form: $f(x)=ax^2+bx+c$, where a, b and c are real numbers, and $a\neq 0$.

Polynomial function: A polynomial in x is an expression of the form $a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_2x^2+a_1x+a_0,\quad a_n\neq 0$, where n is a non-negative integer and the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$ and a_0 are real numbers.

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